GRAPH DIRECTED SELF-CONFORMAL MULTIFRACTALS

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Graph Directed Self-Conformal Multifractals

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Abstract

In this thesis we study the multifractal structure of graph directed self-confromal measures. We begin by introducing a number of notions from geometric measure theory. In particular, several notions of dimension, graph directed iterated function schemes, and the thermodynamic formalism. We then give an historical introduction to multifractal analysis. Finally, we develop our own contribution to multifractal analysis.

Our own contribution to multifractal analysis can be broken into three parts; the proof of two multifractal density theorems, the calculation of the multifractal spectrum of self-conformal measures coded by graph directed iterated function schemes, and the introduction of a relative multifractal formalism together with an investigation of the relative multifractal structure of one graph directed self-conformal measure with respect to another.

Specifically, in Chapter 5 we show that by interpreting the multifractal Hausdorff and packing measures Olsen introduced in [Ol95] as Henstock-Thomson variation measures we are able to obtain two stronger density theorems than those obtained by Olsen.

In Chapter 6 we give full details of the calculation of the multifractal spectrum of graph directed self-conformal measures satisfying the strong open set condition and show that the multifractal Hausdorff and packing measures introduced by Olsen in [Ol95] take positive and finite values at the critical dimension provided that the self-conformal measures satisfy the strong seperation condition.

In Chapter 7 we formalise the idea of performing multifractal analysis with respect to an arbitrary reference measure by developing a formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the 'multifractal formalism' as first introduced by Halsey et. al. [HJKPS86] and closely parallels Olsen's formal treatment of this formalism in [Ol95].

In Chapter 8 we illustrate our relative multifractal formalism by investigating the relative multifractal structure of one graph directed self-conformal measure with respect to another where the two measures are based on the same graph directed self-conformal iterated function scheme which satisfies the strong open set condition.

Declaration

I, Julian Cole, hereby certify that this thesis, which is approximately 31,000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Date 12/1/99 Signature

I was admitted as a research student in September 1995 and as a candidate for the degree of doctor of philosophy in September 1997; the higher study for which this is a record was carried out in the University of St. Andrews between 1995 and 1998.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of doctor of philosophy in the University of St. Andrews and that the candidate is qualified to submit this thesis in the application for that degree.

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Contents

Introduction

The research in this thesis can be divided into three parts; the proof of two multifractal density theorems, the calculation of the multifractal spectrum of self-conformal measures coded by graph directed iterated function schemes, and the introduction of a relative multifractal formalism together with an investigation of the relative multifractal structure of one graph directed self-conformal measure with respect to another. In order to set these results in context we introduce a number of basic concepts from the field of geometric measure theory. In particular we discuss graph directed iterated function schemes, multifractals and the thermodynamic formalism.

This thesis is divided into eight chapters. In outline these chapters deal with the following subjects: fractal measures and dimensions, graph directed iterated function schemes, the thermodynamic formalism, multifractal analysis, multifractal density theorems, the multifractal spectrum of graph directed self-conformal measures satisfying the strong open set condition (an important separation condition in geometric measure theory), a relative multifractal formalism, and the relative multifractal structure of one graph directed self-conformal measure with respect to another.

In the chapter on fractal measures and dimensions we introduce three different notions of dimension of sets, Hausdorff dimension, packing dimension and box-counting dimension. We also consider the measures that accompany these dimensions *i.e.* the Hausdorff and centred Hausdorff measures, and the packing measure. Also included in this chapter are details of the relationship between these measures and dimensions and important covering theorems which are frequently used when calculating these dimensions in particular situations.

While this first chapter can definitely be seen as introductory, in our second chapter we move on to consider one of the most important concepts in fractal geometry, that of an iterated function scheme. The chapter introduces, and covers much of the basic material on, graph directed iterated function schemes. In particular, it discusses invariant sets, symbolic dynamics and invariant measures.

Having introduced one half of what we wish to study *i.e.* graph directed iterated function schemes and invariant measures, we then turn to the other half of our thesis, multifractal analysis. A central theme in the development of multifractal analysis is its intuitive link with statistical mechanics. For this reason, before introducing multifractal analysis, we include a chapter on the thermodynamic formalism. Because we are only interested in using the thermodynamic formalism in the context of the code space, we only introduce it in this restricted setting. In particular, we introduce topological pressure, Gibbs states, and entropy and derive the link between pressure and entropy *i.e.* the variational principle.

Having covered this further introductory material we are free to develop multifractal analysis. The literature on this subject is extensive and it would certainly be possible to write a book in several volumes on it, thus we have restricted our attention to what we believe are the important developments in the field from a historical perspective. In particular, we give a short historical introduction to multifractals and then discuss three papers that we believe have been of special mathematical importance. These papers are [Ra89], [CM92] and [Ol95]. The first of these papers is important because it is among the first mathematically rigourous papers on multifractals, the second because its simple setting and measure theoretic approach to multifractals opened up the field to many geometric measure theorists, and the third because it attempts to perform multifractal analysis of general measures.

In [Ol95] Olsen only required two rather weak multifractal density theorems. In the chapter on density theorems we show that by interpreting the multifractal Hausdorff and packing measures introduced by Olsen in [Ol95] as Henstock-Thomson variation measures we are able to obtain two stronger density theorems. In particular we prove the following. Let μ be a probability measure on $\mathbf{R}^d$, ν be a finite Borel measure on $\mathbf{R}^d$, $q, t \in \mathbf{R}$ and $E \subseteq \mathbf{R}^d$. Set,

$$\overline{d}_{\mu}^{q,t}(x,\nu) = \limsup_{r \searrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} \quad \text{and} \quad \underline{d}_{\mu}^{q,t}(x,\nu) = \liminf_{r \searrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t}.$$

Then provided that $\mathcal{H}_{\mu}^{q,t}(E) < \infty$, where $\mathcal{H}_{\mu}^{q,t}$ denotes the multifractal Hausdorff measure, and $\overline{d}_{\mu}^{q,t}(x,\nu) < \infty$ on E, we have

$$\nu(E) = \int_E \overline{d}_{\mu}^{q,t}(x,\nu)\, d\mathcal{H}_{\mu}^{q,t}(x).$$

Also, provided that $\mathcal{P}_{\mu}^{q,t}(E) < \infty$, where $\mathcal{P}_{\mu}^{q,t}$ denotes the multifractal packing measure, and $\underline{d}_{\mu}^{q,t}(x,\nu) < \infty$ on E, we have

$$\nu(E) = \int_E \underline{d}_{\mu}^{q,t}(x,\nu)\, d\mathcal{P}_{\mu}^{q,t}(x).$$

Chapter six of this thesis contains full details of the calculation of the multifractal spectrum of graph directed self-conformal measures satisfying the strong open set condition and shows that Olsen's multifractal Hausdorff and packing measures are positive and finite at the critical dimensions. The approach used in this calculation is essentially a combination of those found in [Pat97] and [KG92]. It uses the thermodynamic formalism to introduce an auxiliary function which is related to the multifractal spectrum via the Legendre transform. In particular, we have the following. Given $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$, a graph directed self-conformal iterated function scheme (GCIFS) with probabilities based on a strongly connected graph and satisfying the strong open set condition, let $\psi(\omega) = \log |T'_{\omega_1}(\pi(\sigma(\omega)))|$ and $\phi(\omega) = \log p_{\omega_1}$, where σ and π respectively denote the left shift operator and the natural projection map. Also let $P\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $P(q,\beta) = P(q\phi + \beta\psi)$, where $P(\varphi)$ denotes the topological pressure of φ. Finally let us define $\beta : \mathbf{R} \to \mathbf{R}$ using the equation $P(q, \beta(q)) \equiv 0$. Then there exists an interval $(\underline{a}, \overline{a})$ such that for $\alpha \in (\underline{a}, \overline{a})$, $f(\alpha) = F(\alpha) = \beta^*(\alpha)$, where $f(\alpha)$ and $F(\alpha)$ respectively denote the Hausdorff and packing multifractal spectrum of any of the measures generated by the GCIFS and β^* denotes the Legendre transform of β.

In several recent papers (see, for example, [RS1], [LV98] and [Das98]) the idea of performing multifractal analysis with respect to an arbitrary reference measure has been discussed. In chapter seven we formalise these ideas by developing a formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the 'multifractal formalism' as first introduced by Halsey et. al. [HJKPS86] and closely parallels Olsen's formal treatment of this formalism in [Ol95].

In the final chapter of this thesis we illustrate our relative multifractal formalism by investigating the relative multifractal structure of one graph directed self-conformal measure with respect to another where the two measures are based on the same GCIFS which satisfies the strong open set condition. In particular, in addition to the notation introduced for Chapter 6 let $G' = \left(V, E, (T_e)_{e\in E}, (m_e)_{e\in E}\right)$ be a second GCIFS with probabilities based on the same GCIFS as G. Set $\chi(\omega) = \log m_{\omega_1}$ and define $P' : \mathbf{R}^2 \to \mathbf{R}$ by $P'(q,\beta) = P(q\chi + \beta\psi)$. Define $\beta' : \mathbf{R} \to \mathbf{R}$ using the equation $P'(q, \beta'(q)) \equiv 0$ and set $\alpha'(q)$ equal to the minus the derivative of β' at q. Also, define $Q : \mathbf{R}^2 \to \mathbf{R}$ by $Q(q,\zeta) = P(q\phi + \zeta\chi)$, $\zeta_{\mu_u,\nu_u} : \mathbf{R} \to \mathbf{R}$ using the equation $Q(q, \zeta_{\mu_u,\nu_u}(q)) \equiv 0$ and set $\gamma_{\mu_u,\nu_u}(q)$ equal to the minus the derivative of ζ_{μ_u,ν_u} at q. Finally, for $\gamma, \alpha \geq 0$, define the sets $K_u(\gamma)$ and $K_u(\gamma,\alpha)$ in the following way:

$$K_u(\gamma) = \left\{ x \in \operatorname{supp}\mu_u \cap \operatorname{supp}\nu_u \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu_u(B(x,r))}{\log \nu_u(B(x,r))} = \gamma \right\}$$

and

$$K_u(\gamma,\alpha) = \left\{ x \in \operatorname{supp}\mu_u \cap \operatorname{supp}\nu_u \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu_u(B(x,r))}{\log \nu_u(B(x,r))} = \gamma \text{ and } \lim_{r\searrow 0} \frac{\log \nu_u(B(x,r))}{\log r} = \alpha \right\},$$

where μ_u and ν_u respective denote the self-similar measure associated with G and G' at the vertex u. With these definitions we have the following results: There exists an interval $(\underline{c}, \overline{c})$ such that for $\gamma \in (\underline{c}, \overline{c})$,

$$\dim_{\nu_u} K_u(\gamma) = \operatorname{Dim}_{\nu_u} K_u(\gamma) = \zeta^*_{\mu_u,\nu_u}(\gamma)$$

where $\dim_{\nu_u}$ and $\operatorname{Dim}_{\nu_u}$ respectively denote the ν_u-Hausdorff and ν_u-packing dimension and

$$\dim_{\mathrm{H}} K_u(\gamma_{\mu_u,\nu_u}(q), \alpha'(q)) = \dim_{\mathrm{P}} K_u(\gamma_{\mu_u,\nu_u}(q), \alpha'(q)) = \alpha'(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right).$$

Finally we give a counter example to the natural conjecture that

$$\dim_{\mathrm{H}} K_u(\gamma_{\mu_u,\nu_u}) = \dim_{\mathrm{P}} K_u(\gamma_{\mu_u,\nu_u}) = \alpha'(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right).$$

1 Fractal Measures and Dimensions

In this chapter we introduce three specific measures and dimensions. It is these that we will be primarily concerned with in this thesis (we will use the centred ν-Hausdorff measure and ν-packing measure and their respective dimensions in chapters seven and eight but we leave their introduction until then). We also give some theory about the relationship between these measures and dimensions which will prove useful later on. Finally, we develop two important covering theorems which are useful for calculating using these measures. There is extensive literature available on this subject, a good overview can be found in Chapters 2 and 3 of [Fa90] and Chapters 4 and 5 of [Mat95].

1.1 Hausdorff Measure and Dimension

Each of the measures and dimensions that we will be considering are generalisations of our intuitive ideas of volume to general fractional dimensions. We start here by defining the most frequently used generalisation, the Hausdorff measure and dimension.

Definition 1.1 *We call a sequence of sets $(A_i)_i$ a δ-*cover *of A if $A \subseteq \cup_{i=1}^{\infty} A_i$ and $0 < \text{diam}\,(A_i) < \delta$ for each i.*

Now given a subset A of $\mathbf{R}^d$, $s \geq 0$ and $\delta > 0$ we define

$$\mathcal{H}^s_\delta(A) = \inf\left\{ \sum_{i=1}^{\infty} \text{diam}\,(A_i)^s \,\middle|\, (A_i)_{i\in\mathbf{N}} \text{ is a countable } \delta\text{-cover of } A \right\}.$$

We now let $\delta \searrow 0$ to obtain the *Hausdorff s-dimensional outer measure* of A *i.e.*

$$\mathcal{H}^s(A) := \lim_{\delta\searrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A).$$

Lemma 1.2 *$\mathcal{H}^s$ is a Borel regular outer measure on $\mathbf{R}^d$.*

Proof: See Corollary 4.5. in [Mat95]. ∎

Note: Carathéodory's Criterion tells us that this is equivalent to saying that $\mathcal{H}^s$ is a regular metric outer measure.

The restriction of $\mathcal{H}^s$ to $\mathcal{H}^s$-measurable sets is called *Hausdorff s-dimensional measure.*

Given that we are seeking to generalise our intuitive ideas of dimension, it may be useful to take a closer look at the relationships between sets of integer dimension. In doing this one question that we might ask is, what is the area of a one dimensional line or the volume of a square? In each case the answer is zero. Another question might be, what is the length of a square or the area of a cube? The answer in each case is infinity. Thus we see that objects that have higher dimension than the dimension that is related to the measure that we are using have a measure of infinity, and objects with lower dimension a measure of zero. This leads naturally to the following theorem and definition of dimension.

Theorem 1.3 *For $0 \leq s < t \leq \infty$ and $A \subseteq \mathbf{R}^d$;*

1. $\mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0$;
2. $\mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = \infty$.

Proof: See Page 28 in [Fa90]. ∎

A consequence of this theorem is that there is a unique number $\dim_{\mathrm{H}}(A)$ for which

$$s < \dim_{\mathrm{H}}(A) \Rightarrow \mathcal{H}^s(A) = \infty$$

$$t > \dim_{\mathrm{H}}(A) \Rightarrow \mathcal{H}^t(A) = 0.$$

Formally we have:

Definition 1.4 *The* Hausdorff dimension $\dim_{\mathrm{H}}(A)$ *of a set* $A \subseteq \mathbf{R}^d$ *is given by*

$$\begin{aligned}\dim_{\mathrm{H}}(A) &= \sup\{s \mid \mathcal{H}^s(A) > 0\} \\ &= \sup\{s \mid \mathcal{H}^s(A) = \infty\} \\ &= \inf\{s \mid \mathcal{H}^s(A) < \infty\} \\ &= \inf\{s \mid \mathcal{H}^s(A) = 0\}.\end{aligned}$$

Two useful properties of the Hausdorff dimension are that it is monotone and σ-stable *i.e.* given $A \subseteq B \subseteq \mathbf{R}^d$ we have $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}(B) \leq d$ and

$$\dim_{\mathrm{H}}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_i \{\dim_{\mathrm{H}}(A_i) \mid i = 1, 2, \ldots\}.$$

When calculating the Hausdorff dimension of a set it is frequently more difficult to find a lower bound than to find an upper bound. A result that is frequently used in such calculations is Lemma 1.5. Before we can state Lemma 1.5 we require the following definition. Given a probability measure ν the Hausdorff, dimension of ν is given by:

$$\dim_{\mathrm{H}} \nu = \inf_{E:\nu(E)>0} \dim_{\mathrm{H}}(E).$$

Lemma 1.5 *If* ν *is a probability measure on* $\mathbf{R}^d$ *and for* ν*-a.a.* x*,*

$$\lim_{r \searrow 0} \frac{\log \nu(B(x,r))}{\log r} = a,$$

then

$$\dim_{\mathrm{H}} \nu = a.$$

Proof: See [Bil65] ∎

The Hausdorff Dimension of a set can also be found by considering the unique transition point of another important outer measure, the centred Hausdorff measure. This measure is defined as follows.

Definition 1.6 *For* $A \subseteq \mathbf{R}^d$ *and* $0 < \delta < \infty$ *we call* $(B(x_i, r_i))_i$ *a* centred δ-covering *of* A *if the* $B(x_i, r_i)$ *are closed balls such that* $A \subseteq \bigcup_i B(x_i, r_i)$*,* $2r_i \leq \delta$ *and* $x_i \in A$*. Given a set* $A \subseteq \mathbf{R}^d$*,* $s \geq 0$ *and* $\delta > 0$ *we define*

$$\mathcal{C}_\delta^s(A) = \left\{ \sum_{i=1}^{\infty} (2r_i)^s \,\middle|\, (B(x_i, r_i))_i \text{ is a centred } \delta\text{-covering of } A \right\}, \quad A \neq \emptyset;$$

$$\mathcal{C}_\delta^s(\emptyset) = 0 \qquad \text{and} \qquad \mathcal{C}_0^s(A) = \sup_{\delta > 0} \mathcal{C}_\delta^s(A).$$

The centred Hausdorff s-dimensional outer measure *of* A *is given by,*

$$\mathcal{C}^s(A) = \sup_{B \subseteq A} \mathcal{C}_0^s(B).$$

Lemma 1.7 $\mathcal{C}^s$ *is a metric outer measure on* $\mathbf{R}^d$*.*

Proof: See [RT88] ∎

In [RT88], Saint Raymond and Tricot also how that the measures $\mathcal{H}^s$ and $\mathcal{C}^s$ are equivalent. This proves the following theorem.

Theorem 1.8 *For* $A \subseteq \mathbf{R}^d$*,*

$$\begin{aligned}\dim_{\mathrm{H}}(A) &= \sup\{s \mid \mathcal{C}^s(A) > 0\} \\ &= \sup\{s \mid \mathcal{C}^s(A) = \infty\} \\ &= \inf\{s \mid \mathcal{C}^s(A) < \infty\} \\ &= \inf\{s \mid \mathcal{C}^s(A) = 0\}.\end{aligned}$$

1.2 Packing Measure and Dimension

The packing dimension of a set was first introduced by Tricot in [Tr80] and can be arrived at in two ways *i.e.* by considering packing measures or by modifying the upper box counting dimension (see the next section) so that it is σ-stable.

We begin with the following definition.

Definition 1.9 *For $A \subseteq \mathbf{R}^d$ and $0 < \delta < \infty$ we call $(B(x_i, r_i))_i$ a* centred δ-packing *of A if the $B(x_i, r_i)$ are closed disjoint balls such that $2r_i \leq \delta$ and $x_i \in A$. Now let $0 \leq s < \infty$ and set*

$$\mathcal{P}_\delta^s = \sup \left\{ \sum_i (2r_i)^s \,\middle|\, (B(x_i, r_i))_i \text{ forms a centred } \delta\text{-packing of } A \right\};$$

then let

$$\mathcal{P}_0^s(A) := \lim_{\delta \searrow 0} \mathcal{P}_\delta^s(A) = \inf_{\delta > 0} \mathcal{P}_\delta^s(A).$$

Finally, we define the packing s-dimensional outer measure *$\mathcal{P}^s$ by*

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}_0^s(A) \,\middle|\, A = \bigcup_{i=1}^{\infty} A_i \right\}.$$

Lemma 1.10 *$\mathcal{P}^s$ is a Borel regular outer measure on $\mathbf{R}^d$.*

Proof: See Page 62 in [Mat95]. ∎

The restriction of $\mathcal{P}^s$ to $\mathcal{P}^s$-measurable sets is called *Packing s-dimensional measure.*

It can easily be shown that $\mathcal{P}^t(A) = 0$ whenever $\mathcal{P}^s(A) < \infty$ and $0 \leq s < t$, and so we see that $\mathcal{P}^s$ defines a dimension in the same way as $\mathcal{H}^s$. Formally:

Definition 1.11 *The* Packing dimension $\dim_P(A)$ *of a set $A \subseteq \mathbf{R}^d$ is given by*

$$\begin{aligned} \dim_P(A) &= \sup\{s \mid \mathcal{P}^s(A) > 0\} \\ &= \sup\{s \mid \mathcal{P}^s(A) = \infty\} \\ &= \inf\{s \mid \mathcal{P}^s(A) < \infty\} \\ &= \inf\{s \mid \mathcal{P}^s(A) = 0\}. \end{aligned}$$

The packing dimension also has the properties that it is monotone and σ-stable. Finally in this section we give the relationship between the Hausdorff and packing measures.

Theorem 1.12 *For $A \subseteq \mathbf{R}^d$,*

$$\mathcal{H}^s(A) \leq \mathcal{P}^s(A).$$

Proof: See Theorem 5.12 in [Mat95]. ∎

1.3 Box Counting Dimensions

There are many equivalent ways in which these dimensions can be defined. We look at the two definitions that include the *covering and packing numbers* of the set.

Definition 1.13 *Let A be a non-empty bounded subset of $\mathbf{R}^d$. For $0 < \epsilon < \infty$ let $N(A, \epsilon)$ be the smallest number of ϵ-balls needed to cover A* i.e.

$$N(A, \epsilon) = \min \left\{ k \,\middle|\, A \subseteq \bigcup_{i=1}^{k} B(x_i, \epsilon) \quad \text{for some } x_i \in \mathbf{R}^d, \ (i = 1, \ldots, k) \right\}$$

and let $P(A, \epsilon)$ be the greatest number of disjoint ϵ-balls with centres in A i.e.

$$P(A, \epsilon) = \max\{k \mid \exists \text{ disjoint balls } B(x_i, \epsilon) \text{ with } x_i \in A \quad (i = 1, \ldots, k)\}$$

Lemma 1.14 *For $A \subseteq \mathbf{R}^d$,*

$$N(A, 2\epsilon) \le P(A, \epsilon) \le N(A, \epsilon/2).$$

Proof: This follows easily from the definitions. ∎

We now define the *upper and lower box counting dimensions.*

Definition 1.15 *With $N(A, \epsilon)$ defined as above we have,*

$$\overline{\dim}_{\mathrm{B}}(A) = \limsup_{\epsilon \searrow 0} \frac{\log N(A, \epsilon)}{-\log \epsilon}$$

$$\underline{\dim}_{\mathrm{B}}(A) = \liminf_{\epsilon \searrow 0} \frac{\log N(A, \epsilon)}{-\log \epsilon}.$$

If these limits are equal then we define the *box counting dimension* of A, $\dim_{\mathrm{B}}(A)$, to be this common value. The following is a corollary to Lemma 1.14.

Lemma 1.16 *With $P(A, \epsilon)$ defined as above, we have*

$$\overline{\dim}_{\mathrm{B}}(A) = \limsup_{\epsilon \searrow 0} \frac{\log P(A, \epsilon)}{-\log \epsilon}$$

$$\underline{\dim}_{\mathrm{B}}(A) = \liminf_{\epsilon \searrow 0} \frac{\log P(A, \epsilon)}{-\log \epsilon}.$$

Another way of looking at the box counting dimensions is as a modification of the Hausdorff dimension, were we restrict the coverings used in defining Hausdorff measure to balls of the same size. It follows immediately that

$$\dim_{\mathrm{H}} A \le \underline{\dim}_{\mathrm{B}}(A) \le \overline{\dim}_{\mathrm{B}}(A) \le d.$$

Finally, we give the relationship between $\overline{\dim}_{\mathrm{B}}$ and $\dim_{\mathrm{P}}$.

Theorem 1.17 *For $A \subseteq \mathbf{R}^d$,*

$$\dim_{\mathrm{P}}(A) = \inf\left\{ \sup_i \overline{\dim}_{\mathrm{B}}(A_i) \,\middle|\, A = \bigcup_{i=1}^{\infty} A_i, \operatorname{diam}(A_i) < \infty \right\}.$$

Proof: See Section 3.3 in [Fa90]. ∎

1.4 Covering Theorems

We end this short chapter with statements of two important covering theorems, the Vitali covering theorem and the Besicovitch covering theorem. We include these theorems here because they are often invaluable when calculating using the measures and dimensions that we have introduced in this chapter.

Our first step is to give the following important geometric result.

Lemma 1.18 *Let $\mathcal{A}$ be a family of closed balls contained in a bounded subset of $\mathbf{R}^d$. Then there exists a countable or finite subfamily $(B(x_i, r_i))_i$ of $\mathcal{A}$ such that*

1. *$(B(x_i, r_i))_i$ is a pairwise disjoint family.*
2. *$\left(\bigcup_{B \in \mathcal{A}} B\right) \setminus \left(\bigcup_{i=1}^{k} B(x_i, r_i)\right) \subseteq \bigcup_{i=k+1}^{\infty} B(x_i, 5r_i)$ for all k.*

Proof: See Lemma 1.9 in [Fa85]. ∎

We now move on to the *Vitali covering theorem.* A collection of sets $\mathcal{V}$ is called a *Vitali class* for $A \subseteq \mathbf{R}^d$ if for each $x \in A$ and $\delta > 0$ there exists $V \in \mathcal{V}$ such that $x \in V$ and $0 < \operatorname{diam}(V) \le \delta$.

Theorem 1.19

1. *Let A be an $\mathcal{H}^s$-measurable subset of $\mathbf{R}^d$ and let $\mathcal{V}$ be a Vitali class of A. Then there exists a finite or countable disjoint sequence $(V_i)_i$ from $\mathcal{V}$ such that either $\sum_i (\text{diam } (V_i))^s = \infty$ or $\mathcal{H}^s (E \setminus \bigcup_i V_i) = 0$.*

2. *If $\mathcal{H}^s (A) < \infty$ then, given $\epsilon > 0$, we may also require that*

$$\mathcal{H}^s (E) \leq \sum_i (\text{diam } (V_i))^s + \epsilon.$$

Proof: See Theorem 1.10 in [Fa85]. ∎

Finally, we state the *Besicovitch covering theorem.*

Theorem 1.20 *Let $d \in \mathbf{N}$. Then there exists an integer $\zeta \in \mathbf{N}$ which satisfies the following: let $A \subseteq \mathbf{R}^d$ and for each $x \in A$ fix a number $r_x > 0$ such that $\sup_{x \in A} r_x < \infty$. Then there exist ζ countable or finite subfamilies $\mathcal{A}_1, \ldots, \mathcal{A}_\zeta$ of $\{B(x, r_x) \mid x \in A\}$ such that*

1. $A \subseteq \bigcup_i \bigcup_{B \in \mathcal{A}_i} B$

2. *$\mathcal{A}_i$ is a family of disjoint sets.*

Proof: See Theorem 4.2 in [Ol95]. ∎

2 Graph Directed Iterated Function Schemes

One of the most important concepts in the growing field of fractal geometry is that of *self-similarity.* In this chapter we will be looking at this concept and how to construct fractals with this property.

Probably the best way to introduce the concept of self-similarity is by constructing a fractal that exhibits it. One of the best known and most easily constructed fractals is the middle third Cantor set. It can be constructed from the unit interval by an iterated sequence of deletions. This process can be seen in Figure 1. Let us consider how we construct the middle third Cantor set. The set E_0 is the interval

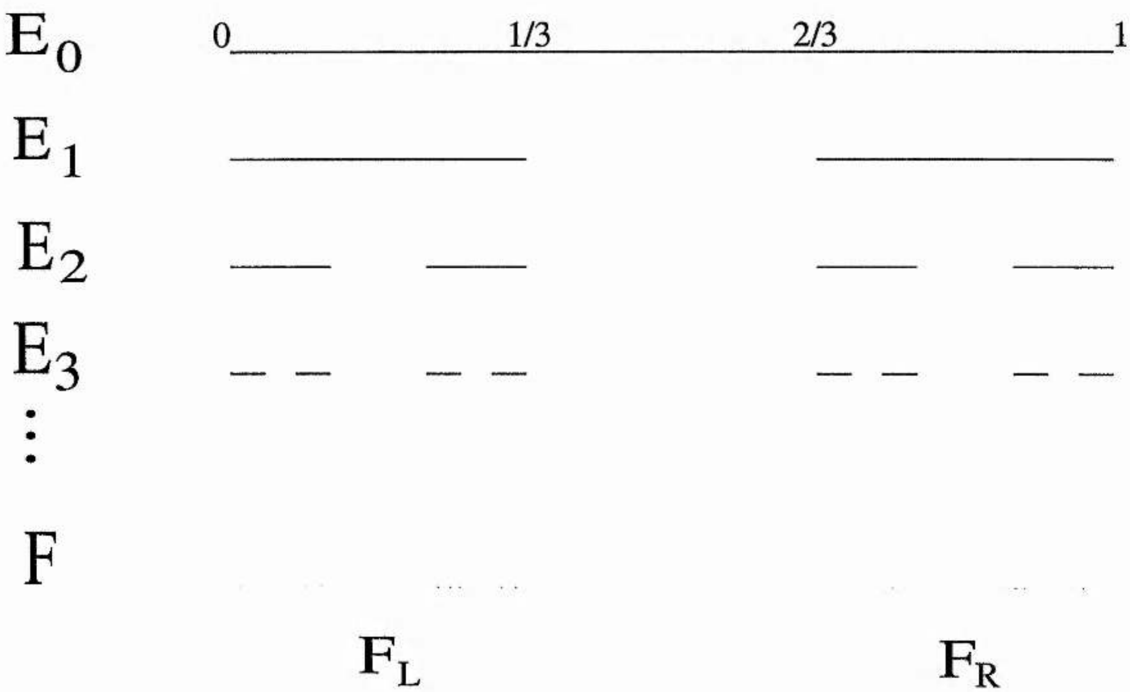


Figure 1: The construction of the middle third Cantor set

$[0,1]$. The set E_1 consists of the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. We obtain E_1 from E_0 by removing the middle third of the interval $[0,1]$ *i.e.* by removing the open interval $(\frac{1}{3}, \frac{2}{3})$. The set E_2 consists of the intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. The set E_2 is obtained from E_1 by removing the middle third of the two intervals in E_1. In fact at each level k we can obtain E_{k+1} from E_k by removing the middle third of each of the intervals in E_k. The *middle third Cantor set* F is the intersection of the sets E_k *i.e.* $F = \bigcap_{k=0}^{\infty} E_k$.

Now if we consider the object F constructed in Figure 1 then we can see that it consists of two parts, marked F_L and F_R. Each of F_L and F_R is a scaled down copy of F. Further, if we consider these two parts F_L and F_R then since each is a scaled down copy of F each of these two parts in turn consists of two scaled down copies of itself. In fact this property of each part consisting of two scaled down versions of itself exists at all scales in the constructed set. It is this property of a set consisting of scaled down copies of itself at all scales that we call self-similarity.

In Figure 2 (overleaf) we have shown how to construct another famous fractal, the *von Koch curve.* Once again we see that it is constructed by a process of iteration and that it also exhibits the property of self-similarity. The four different scaled down copies of the whole are marked F_1, F_2, F_3 and F_4.

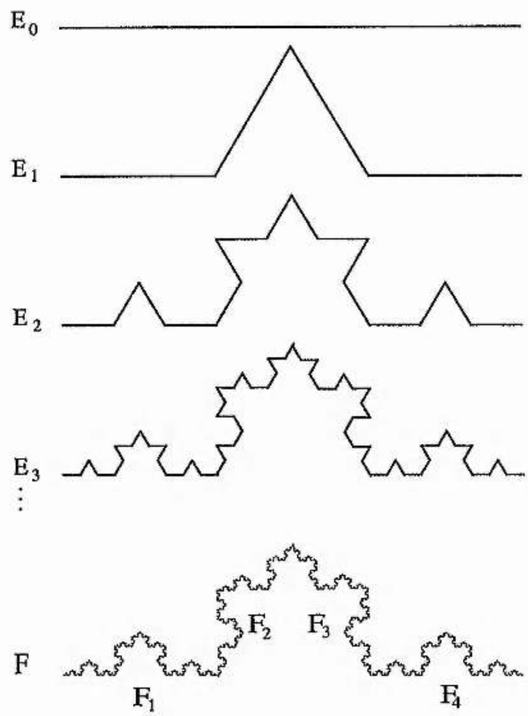


Figure 2: The construction of the von Koch Curve

In some sense we see that the middle third Cantor set and the von Koch curve are defined by the transformations in Figure 3 and Figure 4, respectively.

Figure 3: A generator for the middle third Cantor set

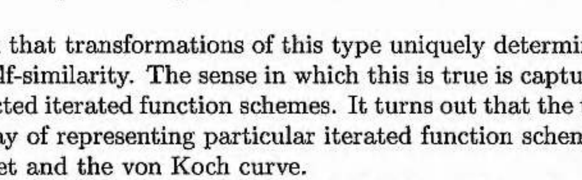


Figure 4: A generator for the von Koch Curve

In fact it is well known that transformations of this type uniquely determine a compact fractal which exhibits the property of self-similarity. The sense in which this is true is captured in the main theorem of this chapter on graph directed iterated function schemes. It turns out that the transformations in Figure 3 and Figure 4 are just a way of representing particular iterated function schemes that respectively define the middle third Cantor set and the von Koch curve.

Much work has been done on this concept of self-similarity and how it is related to a collection of contracting maps. The first real contributor was Moran in his 1946 paper [Mo46]. In this paper Moran worked in a more general setting *i.e.* instead of specifying the construction by maps he merely specified seed sets and contraction ratios and how these should be used to construct a sequence of nested sets whose intersection was the construction object. Moran's main contribution was to derive an expression for the Hausdorff dimension of such a set provided it satisfied a separation condition. More recently, in his 1981 paper [Hu81], Hutchinson formalised a new way of looking at the subclass of these sets which are map specified. He showed that a self-similar set could be viewed as the fixed point of a contraction defined on the class of non-empty, compact subsets of a metric space X. In particular, he proved the following results.

Theorem 2.1

1. *Let $X = (X, d)$ be a complete metric space and let $\mathcal{T} = \{T_1, \dots, T_n\}$ be a finite set of contracting maps on X. Then there exists a unique non-empty closed bounded set K such that $K = \bigcup_{i=1}^{n} T_i(K)$. Further, K is the closure of the set of fixed points of finite compositions $T_{i_1} \circ \dots \circ T_{i_p}$ of members of $\mathcal{T}$. For arbitrary $A \subseteq X$ let $\mathcal{S}(A) = \bigcup_{i=1}^{n} T_i(A)$ and $\mathcal{S}^p(A) = \mathcal{S}(\mathcal{S}^{p-1}(A))$. Then for non-empty closed bounded A, $\mathcal{S}^p(A) \to K$ with respect to the Hausdorff metric.*

2. *In addition to the hypotheses of* (1) *let us suppose that $(p_1, \dots, p_n)$ is a probability vector* i.e. *$p_1, \dots, p_n \in (0,1)$ and $\sum_{i=1}^{n} p_i = 1$. Then there exists a unique Borel regular measure μ of total mass 1 such that $\mu = \sum_{i=1}^{n} p_i \cdot \mu \circ S_i^{-1}$. Furthermore,* $\operatorname{supp} \mu = K$.

3. *Further, if the $T_i \in \mathcal{T}$ are similarities with contraction ratios r_i and are such that they satisfy a separation condition known as the open set condition (see Section 2.2) and s is the unique positive number satisfying $\sum_{i=1}^{n} r_i^s = 1$ then $\dim_{\mathrm{H}} K = s$ and $0 < \mathcal{H}^s(K) < \infty$.*

With the introduction of the packing measure and dimension in the 1980s ([Tr80] and [Tr82]) an interest was also taken in what the packing dimension and measure of these sets would be. Although we are unable to determine exactly when the following theorem appeared it was certainly known by 1990 when it was published in Edgar's book [Ed90].

Theorem 2.2 *Using the notation of Theorem 2.1, if the $T_i \in \mathcal{T}$ are similarities such that they satisfy the open set condition then* $\dim_{\mathrm{P}} K = s$ *and* $0 < \mathcal{P}^s(K) < \infty$.

In fact in [Sp92], Spear was able to show that given $\mathcal{T}$, an iterated function scheme consisting of similarities defined on $\mathbf{R}^d$, and K, the self similar set associated with $\mathcal{T}$, there exists $C_{\mathcal{T}} \in (0, \infty)$, a constant depending on $\mathcal{T}$, such that $\mathcal{H}^s \llcorner K = C_{\mathcal{T}} \mathcal{P}^s \llcorner K$.

In this chapter we will be looking at iterated function schemes coded by directed multigraphs. These are a way of constructing a vector $(K_u)_{u \in V}$ of self-similar sets simultaneously. A definitive formulation of the ideas involved from the perspective of geometric measure theory was given by Mauldin and Williams in their 1988 paper [MW88] but the idea had existed in the literature of dynamical systems previously and had been used by several authors, usually under the name of subshifts of finite type or Markov chains (see [LM95]). In their paper Mauldin and Williams showed that the Hausdorff dimension of these sets could be determined by considering a square matrix with rows and columns indexed by the vertex set of the graph. For each positive number t we define a square matrix $A(t)$ with the entry in row u and column v being:

$$A_{u,v}(t) = \sum_{e \in E_{u,v}} r_e^t,$$

where $E_{u,v}$ is the set of edges from u to v and r_e is the similarity ratio of the map associated with e. The Hausdorff dimension of the set $K = \bigcup_{u \in V} K_u$ is the unique non-negative number s such that the spectral radius of the matrix $A(s)$ is one. Mauldin and Williams also showed that the Hausdorff measure at the critical dimension is positive and either finite or σ-finite depending on the structure of the graph. Soon afterwards it was shown that the packing dimension of the sets K_u coincided with their Hausdorff dimension and that the packing measures were also positive and finite or σ-finite depending on the structure of the graph (see [Ed90]).

In this thesis we will be using a formulation of the idea of graph directed self-similar sets very similar to, but not identical to, that introduced by Mauldin and Williams. Details of the approach that we will use can be found in Edgar's book [Ed90].

We now turn to introducing graph directed self-similar sets, sets whose structure is determined by contractions coded by the edges of directed multigraphs. A system of contractions of this type is called a graph directed iterated function scheme (GDIFS). The remainder of this chapter is spent developing the theory behind GDIFSs which closely parallels the well known theory behind iterated function schemes.

2.1 Graphs

In this section we introduce some definitions/notations for directed graphs. A *finite directed connected (multi)graph* G is an ordered pair (V, E) consisting of a finite set V of vertices and a finite set E of directed edges between vertices, where each pair of vertices is connected together by a path (not necessarily directed). Figure 5 gives a typical example of a finite directed connected graph. In this figure $V = (v_1, \ldots, v_4)$ and $E = (e_1, \ldots, e_{10})$.

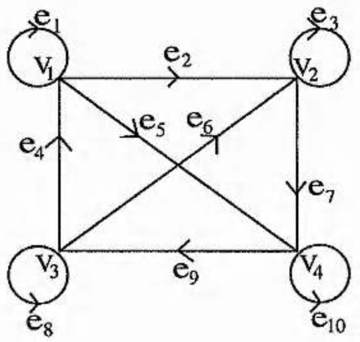


Figure 5: A finite directed connected graph

Let $G = (V, E)$ be a finite directed connected multigraph. For $u, v \in V$ let $E_{u,v}$ denote the set of edges from u to v and set $E_u = \bigcup_{v \in V} E_{u,v}$ *i.e.* the set of all edges leaving u. For $e \in E$ let us denote the initial vertex of e by $i(e)$ and the terminal vertex of e by $t(e)$. We will call a finite string $e_1 e_2 ... e_n$ of edges of G a *finite path* if for $i = \{1, ..., n-1\}$, $t(e_i) = i(e_{i+1})$. Similarly we will call an infinite string $e_1 e_2 ...$ of edges of G an *infinite path* if for all $i \in \mathbf{N}$, $t(e_i) = i(e_{i+1})$. In addition for $u, v \in V$ and $n \in \mathbf{N}$ let us introduce the following notation:

$$\begin{aligned}
E_{u,v}^{(n)} &= \{e_1 ... e_n \mid \text{ paths of length } n \text{ such that } i(e_1) = u \text{ and } t(e_n) = v\}. \\
E_{u,v}^{(*)} &= \bigcup_{n \in \mathbf{N}} E_{u,v}^{(n)} \; \textit{i.e.} \text{ paths of finite length between } u \text{ and } v. \\
E_u^{(n)} &= \bigcup_{v \in V} E_{u,v}^{(n)} \; \textit{i.e.} \text{ paths of length } n \text{ starting at } u. \\
E_u^{(*)} &= \bigcup_{v \in V} E_{u,v}^{(*)} \; \textit{i.e.} \text{ paths of finite length starting at } u. \\
E^{(n)} &= \bigcup_{u \in V} E_u^{(n)} \; \textit{i.e.} \text{ paths of length } n \text{ in } G. \\
E^{(*)} &= \bigcup_{u \in V} E_u^{(*)} \; \textit{i.e.} \text{ paths of finite length in } G. \\
E_u^{\mathbf{N}} &= \{e_1 e_2 ... \mid \text{ infinite paths such that } i(e_1) = u\}. \\
E^{\mathbf{N}} &= \bigcup_{u \in V} E_u^{\mathbf{N}} \; \textit{i.e.} \text{ infinite paths in } G.
\end{aligned}$$

Next we introduce some notation for concatenation and restriction of paths. If $\alpha = \alpha_1 ... \alpha_n \in E^{(*)}$, $\beta = \beta_1 \beta_2 ... \in E^{(*)} \cup E^{\mathbf{N}}$ and $t(\alpha_n) = i(\beta_1)$ then $\alpha\beta = \alpha_1 ... \alpha_n \beta_1 \beta_2 ...$. Also if $\alpha \in E^{(n)}$ then let us write $\mid \alpha \mid = n$. Now if $\alpha \in E^{(n)}$, $\beta \in E^{(m)}$, $n \leq m$ and for $j = \{1, ..., n\}$, $\alpha_j = \beta_j$ then we say $\alpha \leq \beta$. Similarly if $\alpha \in E^{(n)}$, $\omega \in E^{\mathbf{N}}$ and for $j = \{1, ..., n\}$, $\alpha_j = \omega_j$ then we say $\alpha \leq \omega$. Also given $\beta \in E^{(n)}$ or $\beta \in E^{\mathbf{N}}$ and $m \leq n$ or $m \in \mathbf{N}$ respectively then we write $\beta \mid m$ for $\beta_1 ... \beta_m$. For $\alpha \in E^{(*)}$ let us write $t(\alpha)$ for $t\left(\alpha_{|\alpha|}\right)$ and for $\alpha \in E^{\mathbf{N}} \cup E^{(*)}$, $i(\alpha)$ for $i(\alpha_1)$. Finally for $\tau \in E^{(*)}$ we call $[\tau] = \{\omega \in E^{\mathbf{N}} \mid \omega || \tau| = \tau\}$ the *cylinder* associated with τ.

We say that a directed graph is *strongly connected* if given $u, v \in V$, we have that $E_{u,v}^{(*)} \neq \emptyset$. In this thesis we will only consider GDIFSs coded by strongly connected graphs.

2.2 The Invariant Sets

We now introduce iterated functions schemes which are coded by directed multigraphs. Given a vector $(X_v)_{v\in V}$ of complete metric spaces where V is the vertex set of some directed multigraph a GDIFS provides a way of constructing a vector of non-empty compact subsets of these metric spaces. Our first step is to give a formal definition of a GDIFS.

Definition 2.3 *A triple $G = (V, E, (T_e)_{e\in E})$, which consists of a finite directed connected multigraph (V, E) and contracting maps $T_e\colon X_{t(e)} \to X_{i(e)}$ is called a* graph directed iterated function scheme *(GDIFS).*

Given a GDIFS and numbers $p_e \in (0,1)$ such that $\sum_{v\in V}\sum_{e\in E_{u,v}} p_e = 1$ for all $u \in V$ we call the quadruple $G = (V, E, (T_e)_{e\in E}, (p_e)_{e\in E})$ a *GDIFS with probabilities.* A vector $(J_u)_{u\in V}$ of non-empty compact subsets of X_u is called a vector of *seed sets* for a GDIFS $G = (V, E, (T_e)_{e\in E})$ if the sets $(J_u)_{u\in V}$ are regular *i.e.* $\overline{\operatorname{int} J_u} = J_u$, and for each $e \in E$ satisfy

$$T_e\left(J_{t(e)}\right) \subseteq J_{i(e)}. \tag{1}$$

Also given $\tau \in E^{(n)}$ let us set $T_\tau = T_{\tau_1} \circ \cdots \circ T_{\tau_{|\tau|}}$ and given $A \subseteq X_{t(\tau)}$ let us set $A_\tau = T_\tau(A)$.

Having made the requisite definitions we can now go on to establish the existence and uniqueness of a vector $(K_u)_{u\in V}$ of non-empty compact sets which are invariant for the GDIFS in the sense of Equation 2.

Theorem 2.4 *Let $G = (V, E, (T_e)_{e\in E})$ be a GDIFS and let $(J_u)_{u\in V}$ be a vector of seed sets for G. Then there exists a unique vector $K = (K_u)_{u\in V}$ of non-empty compact subsets of X_u satisfying*

$$K_u = \bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(K_v). \tag{2}$$

Further,

$$K_u = \bigcap_{n\in\mathbb{N}}\bigcup_{\tau\in E_u^{(n)}} J_\tau. \tag{3}$$

Proof: For $u \in V$ let $\mathcal{K}_u = (\mathcal{K}(X_u), d_u)$ denote the metric space of compact subsets of X_u together with the Hausdorff metric. The space $\mathcal{K}_u$ is complete, hence, if we set $d = \max_{u\in V} d_u$, then $\mathcal{K} = \left(\prod_{u\in V}\mathcal{K}_u, d\right)$ is complete.

Now let us define $M\colon \mathcal{K} \to \mathcal{K}$ by

$$M\left[(A_u)_{u\in V}\right] = \left(\bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(A_u)\right)_{u\in V}.$$

Equation 2 will follow from the first part of the contraction mapping theorem if we can show that M is a contraction.

For $e \in E$ let $\operatorname{Lip}(T_e)$ denote the Lipschitz ratio of T_e. Then we have,

$$
\begin{aligned}
d\left(M\left[(A_u)_{u\in V}\right], M\left[(B_u)_{u\in V}\right]\right) &= d\left[\left(\bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(A_u)\right)_{u\in V}, \left(\bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(B_u)\right)_{u\in V}\right] \\
&= \max_{u\in V} d_u\left[\left(\bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(A_u)\right), \left(\bigcup_{v\in V}\bigcup_{e\in E_{u,v}} T_e(B_u)\right)\right] \\
&\leq \max_{u\in V}\max_{v\in V}\max_{e\in E_{u,v}} d_u\left[T_e\left(A_v\right), T_e\left(B_v\right)\right] \\
&\leq \max_{e\in E} \operatorname{Lip}\left(T_e\right) \max_{v\in V} d_v\left[A_v, B_v\right] \\
&= \max_{e\in E} \operatorname{Lip}\left(T_e\right) d\left[(A_v)_{v\in V}, (B_v)_{v\in V}\right].
\end{aligned}
$$

Now since $\max_{e\in E} \operatorname{Lip}(T_e) < 1$ we have that M is a contraction.

Equation 3 follows from the second part of the contraction mapping theorem if we iterate the seed sets $(J_u)_{u\in V}$. To see this we note that for each $u \in V$ and $n \in \mathbb{N}$ Equation 1 implies that $\bigcup_{\tau\in E_u^{(n+1)}} J_\tau \subseteq \bigcup_{\tau\in E_u^{(n)}} J_\tau$. ∎

We call the sets K_u the *graph directed self-similar sets* associated with G. It is worth noting that any collection $(J_u)_{u\in V}$ of non-empty compact sets will tend to $(K_u)_{u\in V}$ in the d-metric.

We now give an example of sets generated by a GDIFS. The sets in Figure 6 are based on a GDIFS which we represent pictorially in Figure 7.

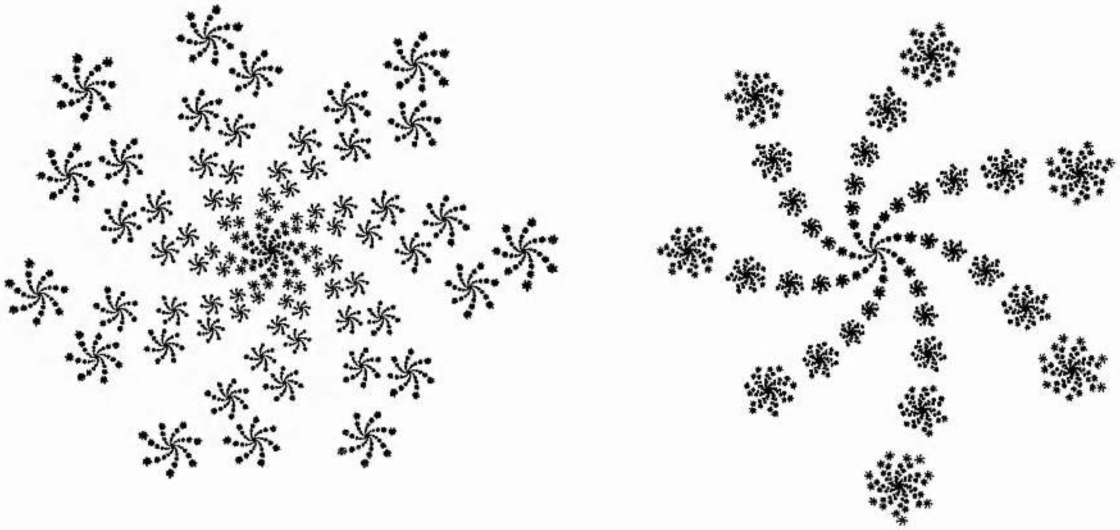
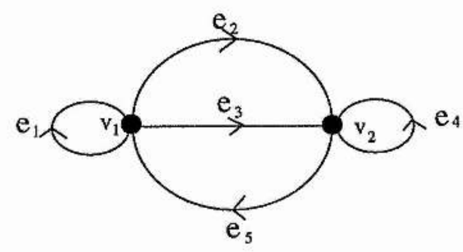


Figure 6: The invariant sets of the GDIFS in Figure 7



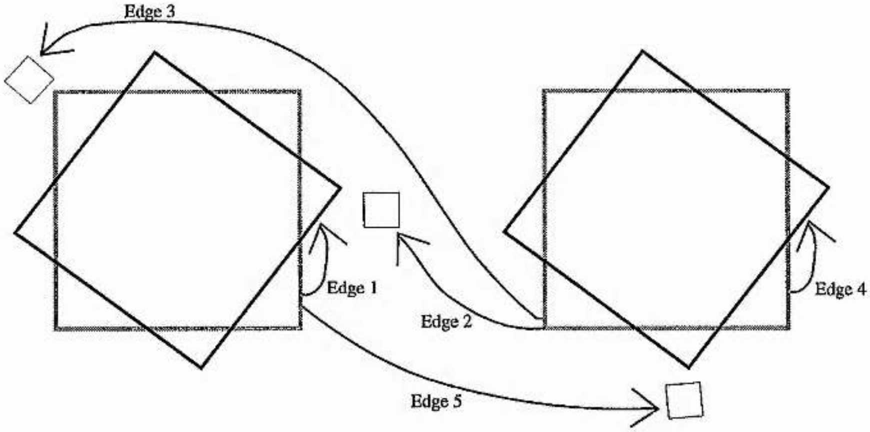


Figure 7: A pictorial representation of the GDIFS for the sets in Figure 6

We end this section by showing that any set that can be defined by an iterated function scheme can also be defined by a GDIFS. Let $\mathcal{T} = \{T_1, \dots, T_n\}$ be an iterated function scheme on X then the GDIFS specified by Figure 8 is a GDIFS defining the same set as $\mathcal{T}$. This is easily seen since the GDIFS generates a non-empty compact subset of X which is invariant under the same maps as the invariant set of $\mathcal{T}$.

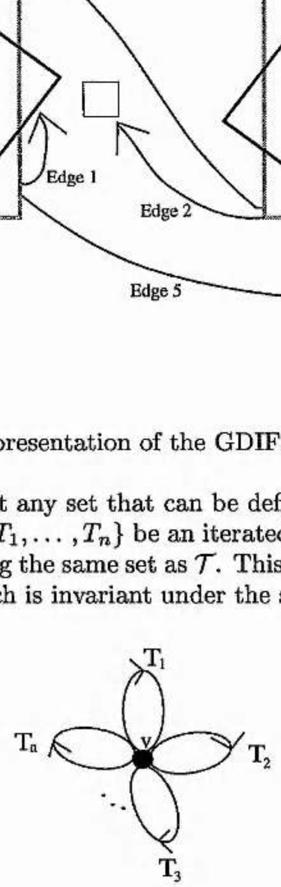


Figure 8: A GDIFS which is equivalent to the iterated function scheme $\mathcal{T} = \{T_1, \dots, T_n\}$

2.3 The Code Space

When investigating the sets associated with a GDIFS we will frequently use symbolic dynamics. When considering GDIFSs the code space that we use is the space $E^{\mathbb{N}}$ as defined in Section 2.1.

Let $\omega \in E_u^{\mathbb{N}}$, then since $\left(T_{\omega|n}\left(K_{t(\omega_n)}\right)\right)_n$ is a decreasing sequence of non-empty compact sets such that $\operatorname{diam}\left(T_{\omega|n}\left(K_{t(\omega_n)}\right)\right) \searrow 0$, $\bigcap_n \left(T_{\omega|n}\left(K_{t(\omega_n)}\right)\right)$ is a singleton. Thus for each $u \in V$ we are able to define a 'code map' or 'projection' $\pi_u\colon E_u^{\mathbf{N}} \to X_u$ by

$$\{\pi_u(\omega)\} = \bigcap_n \left(T_{\omega|n}\left(K_{t(\omega_n)}\right)\right).$$

If $x = \pi_u(\omega) \in E_u$ then we say that ω is an *address* of the point x.

Since the element of $E^{\mathbf{N}}$ that the code map is projecting uniquely defines a $u \in V$ for notational convenience we will sometimes drop this subscript *i.e.* denote π_u by π.

An important property of the code maps is that for each $e \in E$ and $\omega \in E_{t(e)}^{\mathbf{N}}$,

$$\pi_{i(e)}(e\omega) = T_e(\pi_{t(e)}(\omega)). \tag{4}$$

Equation 4 can easily be proven. For,

$$T_e\left(\pi_{t(e)}(\omega)\right) = T_e\left(\bigcap_{n\in\mathbf{N}} K_{t(\omega|n)}\right) \subseteq \bigcap_{n\in\mathbf{N}} T_e(K_{t(\omega|n)}) = \bigcap_{n\in\mathbf{N}} K_{t(e\omega|n)} = \left\{\pi_{i(e)}(e\omega)\right\}.$$

To aid our investigation of the code space we introduce the following important shift maps, $\sigma\colon E^{\mathbf{N}} \to E^{\mathbf{N}}$ given by $\sigma(\omega_1\omega_2\ldots) = \omega_2\omega_3\ldots$ and for $\tau \in E^{(*)}$, $\sigma_\tau\colon E^{\mathbf{N}} \to E^{\mathbf{N}}$ given by $\sigma_\tau(\omega) = \tau\omega$.

Equation 4 can now be seen to show that the following diagram commutes.

$$\begin{array}{ccc} E^{\mathbf{N}} & \xrightarrow{\sigma_e} & E^{\mathbf{N}} \\ \downarrow{\scriptstyle \pi_{t(e)}} & & \downarrow{\scriptstyle \pi_{i(e)}} \\ K_{t(e)} & \xrightarrow{T_e} & K_{i(e)} \end{array}$$

Thus the dynamical systems $\left(\bigcup_{u\in V} K_u, T_e\right)$ and $\left(E^{\mathbf{N}}, \sigma_e\right)$ are conjugate.

We now use Equation 4 to prove the following important lemma:

Lemma 2.5 *Let $G = (V, E, (T_e)_{e\in E})$ be a GDIFS, $K = (K_u)_{u\in V}$ be the graph directed sets associated with G and $(\pi_u)_{u\in V}$ be the code maps associated with G. Then for each $u \in V$ we have*

$$K_u = \pi_u(E_u^{\mathbf{N}}).$$

Proof: Since $\pi_u(E_u^{\mathbf{N}})$ is both non-empty and compact, being the image under a continuous map of a non-empty compact set, the result will follow from the uniqueness part of Theorem 2.4 if we can show that $\pi_u(E_u^{\mathbf{N}})$ is invariant under M. Now,

$$\begin{aligned}
\bigcup_{e\in E_u} T_e\left(\pi_{t(e)}\left(E_{t(e)}^{\mathbf{N}}\right)\right) &= \bigcup_{e\in E_u} T_e\left(\bigcup_{\omega\in E_{t(e)}^{\mathbf{N}}} \pi_{t(e)}(\omega)\right) \\
&= \bigcup_{e\in E_u}\bigcup_{\omega\in E_{t(e)}^{\mathbf{N}}} T_e\left(\pi_{t(e)}(\omega)\right) \\
&= \bigcup_{e\in E_u}\bigcup_{\omega\in E_{t(e)}^{\mathbf{N}}} \pi_u(e\omega) \\
&= \bigcup_{\tau\in E_u^{\mathbf{N}}} \pi_u(\tau). \\
&= \pi_u(E_u^{\mathbf{N}}).
\end{aligned}$$

■

Next we make the code space $E^{\mathbf{N}}$ into a metric space by defining the following metric d on it. Let $r_{\max} = \max_{e\in E} \mathrm{Lip}\,(T_e)$. For $\omega, \tau \in E^{\mathbf{N}}$ let $d[\omega,\tau] = r_{\max}^n$, where $n = \min\{i \in \mathbf{N} \mid \omega_i \neq \tau_i\}$. Using this definition we prove the following lemma about the projections.

Lemma 2.6 *Given a GDIFS defined on the vector of metric spaces $(X_u, \rho_u)_{u\in V}$ we have that for each $u \in V$, π_u is Lipschitz.*

Proof: Given $\tau, \omega \in E_u^{\mathbf{N}}$ if we set $n = \min\{i \in \mathbf{N} \mid \omega_i \neq \tau_i\}$ we have

$$\begin{array}{rcl} \rho_u[\pi_u(\tau), \pi_u(\omega)] & \leq & \operatorname{diam} T_{\omega|n-1}(K_{t(\omega_{n-1})}) \\ & \leq & r_{\max}^{n-1} \max_u \operatorname{diam} K_u \\ & = & Cd[\omega, \tau] \end{array}$$

where $C = \max_u \operatorname{diam} K_u / r_{\max}$. ■

Note: We note that later on we will define an equivalent metric on $E^{\mathbf{N}}$ given by $d[\tau, \omega] = r_{\max}^{\gamma n}$ for some $\gamma \in (0,1)$. Lemma 2.6 still holds with this metric.

2.4 The Invariant Measures

In this section we show that given a GDIFS with probabilities we can associate a vector $(\mu_u)_{u\in V}$ of probability measures with it in a natural way.

First, we introduce the Hutchison metric L on $\mathcal{M}^1(X)$, the space of probability measures on X. For $\mu, \nu \in \mathcal{M}^1(X)$ let

$$L(\mu, \nu) = \sup \left\{ \left| \int \phi d\mu - \int \phi d\nu \right| \phi \colon X \to \mathbf{R},\ \operatorname{Lip} \phi \leq 1 \right\}$$

and note that it generates the weak topology on $\mathcal{M}^1(X)$.

Recall that a GDIFS with probabilities is a list $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$, where

1. $\left(V, E, (T_e)_{e\in E}\right)$ is a GDIFS.
2. for $e \in E$, $p_e \in (0,1)$ and for all $u \in V$,

$$\sum_{v\in V} \sum_{e\in E_{u,v}} p_e = 1$$

Given $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$, a GDIFS with probabilities, define $\Psi\colon \prod_{u\in V} \left(\mathcal{M}^1(X_u)\right) \to \prod_{u\in V} \left(\mathcal{M}^1(X_u)\right)$ by

$$\Psi\left((\nu_u)_{u\in V}\right) = \left(\sum_{e\in E_u} p_e.\nu_{t(e)} \circ T_e^{-1}\right)_{u\in V}.$$

We say that a vector of measures $(\nu_u)_{u\in V}$ is *G-self-similar* if,

$$\Psi\left((\nu_u)_{u\in V}\right) = (\nu_u)_{u\in V}.$$

Also given $(\nu_u)_{u\in V} \in \prod_{u\in V} \mathcal{M}^1(X_u)$ let

$$\Psi^0\left((\nu_u)_{u\in V}\right) = \left(\sum_{e\in E_u} p_e.\nu_{t(e)} \circ T_e^{-1}\right)_{u\in V} \quad \text{and } \Psi^k\left((\nu_u)_{u\in V}\right) = \Psi\left(\Psi^{k-1}\left((\nu_u)_{u\in V}\right)\right).$$

If for each $u \in V$ we equip $\mathcal{M}^1(X_u)$ with the Hutchinson metric as defined above and denote it by L_u then we are able to define a metric L on $\prod_{u\in V} \left(\mathcal{M}^1(X_u)\right)$ by setting

$$L\left[(\nu_u)_{u\in V}, (\mu_u)_{u\in V}\right] = \max_{u\in V} L_u(\nu_u, \mu_u).$$

Now the L metric induces the product of the weak topologies and the space $\mathcal{P} = \left(\prod_{u\in V} \mathcal{M}^1(X_u), L\right)$ is complete, thus we can apply the contraction mapping theorem to obtain the following theorem.

Theorem 2.7 *Let $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$ be a GDIFS with probabilities. Then there exists a unique vector $(\mu_u)_{u\in V} \in \mathcal{P}$ of measures such that $(\mu_u)_{u\in V}$ is G-self-similar. Further, if $(\nu_u)_{u\in V} \in \mathcal{P}$ then $\Psi^k\left((\nu_u)_{u\in V}\right) \to (\mu_u)_{u\in V}$ with respect to the product of the weak topologies.*

Proof: We only require to show that Ψ is a contraction on $\mathcal{P}$. Now if we set $\mu(\phi) = \int \phi d\mu$ and $r_{\max} = \max_{e\in E} r_e$ then for $u \in V$, $e \in E_u$ and each family of functions $(\phi_u : X_u \to \mathbf{R})_{u\in V}$ such that $\operatorname{Lip}\phi_u \le 1$ for each u we have that $\operatorname{Lip}\left(r_{\max}^{-1}\phi_{t(e)} \circ T_e^{-1}\right) \le 1$. Thus for $(\nu_u)_{u\in V}, (\mu_u)_{u\in V} \in \mathcal{P}$ we have

$$
\begin{aligned}
L\left[\Psi\left((\mu_u)\right)_{u\in V}, \Psi\left((\nu_u)\right)_{u\in V}\right] &= \max_{u\in V}\left| \sum_{e\in E_u} p_e \cdot \left(\mu_{t(e)} \circ T_e^{-1}\right)\left(\phi_{t(e)}\right) - \sum_{e\in E_u} p_e \cdot \left(\nu_{t(e)} \circ T_e^{-1}\right)\left(\phi_{t(e)}\right)\right| \\
&= \max_{u\in V}\left| \sum_{e\in E_u} p_e \cdot \left(\mu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right) - \nu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right)\right)\right| \\
&= \max_{u\in V}\left| \sum_{e\in E_u} p_e r_{\max}\left(r_{\max}^{-1}\mu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right) - r_{\max}^{-1}\nu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right)\right)\right| \\
&\le \max_{u\in V} \sum_{e\in E_u} p_e r_{\max}\left| \left(r_{\max}^{-1}\mu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right) - r_{\max}^{-1}\nu_{t(e)}\left(\phi_{t(e)} \circ T_e^{-1}\right)\right)\right| \\
&\le r_{\max} \max_{u\in V} \sum_{e\in E_u} p_e L_{t(e)}\left[\mu_{t(e)}, \nu_{t(e)}\right] \\
&\le r_{\max} \max_{u\in V} L_u\left[\mu_u, \nu_u\right] \\
&= r_{\max} L\left[(\nu_u)_{u\in V}, (\mu_u)_{u\in V}\right].
\end{aligned}
$$

■

We now derive an alternative expression for these self-similar measures. We do this by using Kolmogorov's consistency theorem to define an infinite probability measure $\hat{\mu}_u$ on $E_u^{\mathbf{N}}$ such that $\hat{\mu}_u([\tau]) = p_\tau$ for all $\tau \in E_u^{(*)}$, where $p_\tau := p_{\tau_1} \cdots p_{\tau_{|\tau|}}$. With $\hat{\mu}_u$ defined in this way we have the following lemma.

Lemma 2.8 *Let $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$ be a GDIFS with probabilities and let $(K_u)_{u\in V}$ be the unique non-empty compact invariant sets associated with G. In addition, for each $u \in V$ let $\hat{\mu}_u$ be defined as above and let π_u be the projection map from $E_u^{\mathbf{N}} \to K_u$. Finally, let $(\mu_u)_{u\in V}$ be the unique G-self-similar probability measures. Then,*

1. $\mu_u = \hat{\mu}_u \circ \pi_u^{-1}$;
2. $\operatorname{supp}\mu_u = K_u$.

Proof:
(1) We have that for each $e \in E_u$, $\pi_u \circ \sigma_e = T_e \circ \pi_{t(e)}$ and that $\sum_{e\in E_u} p_e \cdot \hat{\mu}_{t(e)} \circ \sigma_e^{-1} = \hat{\mu}_u$. Thus,

$$
\begin{aligned}
\sum_{e\in E_u} p_e \cdot \left(\left(\hat{\mu}_{t(e)} \circ \pi_{t(e)}^{-1}\right) \circ T_e^{-1}\right) &= \sum_{e\in E_u} p_e \cdot \left(\left(\hat{\mu}_{t(e)} \circ \sigma_e^{-1}\right) \circ \pi_u^{-1}\right) \\
&= \left(\sum_e p_e \cdot \hat{\mu}_{t(e)} \circ \sigma_e^{-1}\right) \circ \pi_u^{-1} \\
&= \hat{\mu}_u \circ \pi_u^{-1}.
\end{aligned}
$$

The result now follows from the uniqueness of μ_u.
(2) Follows immediately from part (1) and Lemma 2.5. ■
The important implication of this theorem is that, provided the GDIFS satisfies the SSC, for each $\tau \in E_u^{(*)}$, $\mu_u(K_\tau) = p_\tau$. In fact we will see that if a GDIFS satisfies the strong open set condition then this is still true (see Chapter 6).

3 The Thermodynamic Formalism

Bowen and Ruelle's thermodynamic formalism is frequently used when calculating the multifractal spectrum of measures. In this chapter we introduce parts of this thermodynamic formalism in the setting of the code space. In particular we introduce Gibbs states and derive the variational principle. Some of the definitions we use are not those that can be found in the more general setting, but definitions that are equivalent in our simple setting. For readers who are interested in the more general theory a good overview can be found in [Ru78].

3.1 Mathematical Preliminaries

We start our work on the thermodynamic formalism with some elementary facts about sequences. Proofs of these results can be found in [Fa97]. A sequence $(a_n)_n$ of real numbers is called *subadditive* if for each $n, m \in \mathbf{N}$,

$$a_{n+m} \leq a_n + a_m.$$

Lemma 3.1 *Let $(a_n)_n$ be a subadditive sequence of real numbers. In this situation $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_{n\in\mathbf{N}} a_n/n$.*

Lemma 3.2 *Let $p_1, \ldots, p_n$ be a probability vector* i.e. $\sum_{i=1}^{n} p_i = 1$ *and $p_i \in (0,1)$ for each i. Also let $q_1, \ldots, q_n$ be real numbers. Then,*

$$\sum_{i=1}^{m} p_i \left(-\log p_i + q_i\right) \leq \log\left(\sum_{i=1}^{n} e^{q_i}\right).$$

We end this section with the other important theorem which we require in this chapter, *Schauder's Fixed Point Theorem.* A proof of this theorem can be found in [DS63].

Theorem 3.3 *Let E be a non-empty compact convex subset of a locally convex topological vector space. Then any continuous function $f\colon E \to E$ has a fixed point.*

3.2 Bounded Variation

The celebrated principle of bounded variation plays an important role in the thermodynamic formalism. Consideration shows that it is the fundamental result which allows all others to follow. In this section we show that a Hölder continuous function defined on the code space satisfies the principle of bounded variation. In chapter six, we will introduce graph directed self-conformal iterated function schemes (GCIFSs). An important feature of these is that there exists a number $r_{\max} < 1$ such that $r_{\max}$ is the largest contraction ratio of the GCIFS and a number $\gamma \in (0,1)$ such that each of the maps in the GCIFS is γ-Hölder continuous. We now use $r_{\max}$ and γ to define a metric on the code space. We start by observing that if we define $c_{\max} = r_{\max}^{\gamma}$ then $c_{\max} < 1$. Now given $\tau, \omega \in E^{\mathbf{N}}$ we set $d[\tau, \omega] = c_{\max}^{n}$, where $n = \min\{i \mid \tau_i \neq \omega_i\}$.

Let $\phi\colon E^{\mathbf{N}} \to \mathbf{R}$ be γ-Hölder Continuous i.e there exists $C \in (0,\infty)$ such that for all $\tau, \omega \in E^{\mathbf{N}}$, $|\phi(\tau) - \phi(\omega)| \leq C\, d[\tau,\omega]^{\gamma}$. Set $S_n\phi(\omega) = \sum_{i=0}^{n-1} \phi\left(\sigma^i(\omega)\right)$. With these definitions we have the following theorem, known as the *Principle of Bounded Variation.*

Theorem 3.4 *Let $\phi\colon E^{\mathbf{N}} \to \mathbf{R}$ be γ-Hölder Continuous, then there exists $b \in (0,\infty)$ such that for all $n \in \mathbf{N}$, $\tau \in E^{(n)}$ and $\omega, \alpha \in [\tau]$,*

$$|S_n\phi(\omega) - S_n\phi(\alpha)| \leq b$$

or equivalently,

$$e^{-b} \leq \frac{\exp\left(S_n\phi(\omega)\right)}{\exp\left(S_n\phi(\alpha)\right)} \leq e^{b}. \tag{5}$$

Proof: Let $\omega, \alpha \in E^{\mathbf{N}}$, then there exists $C \in (0, \infty)$ such that

$$\begin{aligned}|\phi\left(\sigma^i(\omega)\right) - \phi\left(\sigma^i(\alpha)\right)| &\leq C\ d[\sigma^i(\omega), \sigma^i(\alpha)]^\gamma \\ &\leq C\ c_{\max}^{i\gamma}\ d[\omega, \alpha]^\gamma.\end{aligned}$$

Thus we have

$$\begin{aligned}|S_n\phi(\omega) - S_n\phi(\alpha)| &\leq \left|\sum_{i=0}^{n-1} \phi\left(\sigma^i(\omega)\right) - \sum_{i=0}^{n-1} \phi\left(\sigma^i(\alpha)\right)\right| \\ &\leq \sum_{i=0}^{n-1} \left|\phi\left(\sigma^i(\omega)\right) - \phi\left(\sigma^i(\alpha)\right)\right| \\ &\leq \sum_{i=0}^{n-1} C\ c_{\max}^{i\gamma}\ d[\omega, \alpha]^\gamma \\ &\leq C \frac{c_{\max}^\gamma}{1 - c_{\max}^\gamma} d[\omega, \alpha]^\gamma \\ &\leq C \frac{c_{\max}^\gamma}{1 - c_{\max}^\gamma} \\ &:= b.\end{aligned}$$

■

3.3 Gibbs States and Topological Pressure

We now introduce Gibbs states and topological pressure. The *Gibbs state* of a Hölder continuous function ϕ is the unique σ-invariant probability measure $\tilde{\mu}_\phi$ on $E^{\mathbf{N}}$ such that there exist positive and finite constants a_0 and P satisfying

$$a_0^{-1} \exp(S_n\phi(\omega) - nP) \leq \tilde{\mu}_\phi([\omega|n]) \leq a_0 \exp(S_n\phi(\omega) - nP) \tag{6}$$

for all $\omega \in E^{\mathbf{N}}$ and $n \in \mathbf{N}$. We note that $P = P(\phi)$ is a constant depending on ϕ and is known as the *(topological) pressure* of ϕ. Now for $\tau \in E^{(*)}$ let us choose $\omega_\tau \in [\tau]$ and set $S_{|\tau|}\phi(\tau) = S_{|\tau|}\phi(\omega_\tau)$. The condition that $\tilde{\mu}_\phi$ be a probability measure implies that

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\tau \in E^{(n)}} \exp(S_n\phi(\tau)). \tag{7}$$

Our aim is to show that a Gibbs state measure, as defined by Equation 6, exists and is unique. We will closely follow the treatment given by Falconer in [Fa97]. In order to prove the existence of Gibbs States we require the following operator, known as the *transfer operator*. Let $\phi \colon E^{\mathbf{N}} \to \mathbf{R}$ be γ-Hölder continuous and let $C\left(E^{\mathbf{N}}\right)$ denote the space of real valued continuous functions on $E^{\mathbf{N}}$. Then $L_\phi \colon C\left(E^{\mathbf{N}}\right) \to C\left(E^{\mathbf{N}}\right)$ is defined by

$$\left(L_\phi g\right)(\omega) = \sum_{e, t(e) = i(\omega)} g(e\omega)\, e^{\phi(e\omega)}.$$

This definition is equivalent to

$$\left(L_\phi g\right)(\omega) = \sum_{\alpha, \sigma(\alpha) = \omega} g(\alpha)\, e^{\phi(\alpha)}.$$

The following elementary properties of the transfer operator are easily verified.

Lemma 3.5

1. *The transfer operator is positive and linear.*
2. $\left(L_\phi^n g\right)(\omega) = \sum_{\tau \in E^{(n)}, t(\tau)=i(\omega)} g(\tau\omega) \exp\left(S_n \phi(\tau\omega)\right)$.
3. $\left(L_\phi\left((g_1 \circ \sigma) \times g_2\right)\right)(\omega) = g_1(\omega)\left(L_\phi g_2\right)(\omega)$.

The next theorem is a special case of the celebrated *Ruelle-Perron-Frobenius theorem*, an extension of the Perron-Frobenius theorem and it shows the importance of the transfer operator in defining Gibbs states.

Theorem 3.6 *Let L_ϕ be defined as above. Then we have:*

1. *There exists a strictly positive eigenvalue λ and positive eigenfunction h associated with λ such that*
$$L_\phi h = \lambda h.$$
2. *There exists a Borel probability measure $\hat{\mu}$ supported on $E^{\mathbb{N}}$ such that for all $g \in C\left(E^{\mathbb{N}}\right)$,*
$$\int \left(L_\phi g\right) d\hat{\mu} = \lambda \int g d\hat{\mu}. \tag{8}$$
3. *Define a probability measure $\hat{\nu}$ supported on $E^{\mathbb{N}}$ by*
$$\int g d\hat{\nu} = \int g \overline{h} d\hat{\mu} \tag{9}$$
for all $g \in C\left(E^{\mathbb{N}}\right)$, where $\overline{h} = h / \int h d\hat{\mu}$. Then $\hat{\nu}$ is σ-invariant.

Proof:
(1) Our first step is to define B, a convex and equicontinuous subset of $C\left(E^{\mathbb{N}}\right)$. To do this we start by observing that there exists $c \in (0, \infty)$ such that for all $e \in E$ and $\omega, \tau \in E^{\mathbb{N}}$ with $t(e) = i(\omega) = i(\tau)$ we have

$$e^{|\phi(e\omega) - \phi(e\tau)|} \leq e^{cd[\omega,\tau]^\gamma}.$$

We also note that we can choose $\alpha > 0$ large enough to satisfy, $\alpha\, c_{\max}^\gamma + c \leq \alpha$. Now let $\beta = e^\alpha > 0$ and define $B = A \cap A'$ where

$$A = \left\{g \in C\left(E^{\mathbb{N}}\right) \mid \beta \leq g(\omega) \leq 1\right\} \text{ and}$$

$$A' = \left\{g \in C\left(E^{\mathbb{N}}\right) \mid \forall u \in V \text{ and } \forall \omega, \tau \in E_u^{\mathbb{N}},\ g(\omega) \leq g(\tau) e^{\alpha d[\omega,\tau]^\gamma}\right\}.$$

It is easy to verify that B is a convex and equicontinuous subset of $C\left(E^{\mathbb{N}}\right)$, thus the Arzela-Ascoli theorem gives us that B is a $||.||_\infty$-compact subset of $C\left(E^{\mathbb{N}}\right)$.

We now define T_ϕ on B, a normalised version of L_ϕ, and show that it maps B into itself and thus, by the Schauder fixed point theorem, has a fixed point. Let $T_\phi g(\omega) = L_\phi g(\omega) / ||L_\phi g||_\infty$ for all $g \in B$. Now all $g \in B$ satisfy

$$\begin{aligned}
\left(L_\phi g\right)(\omega) &= \sum_{e, t(e)=i(\omega)} g(e\omega) e^{\phi(e\omega)} \\
&\leq \sum_{e, t(e)=i(\omega)} g(e\tau) e^{\alpha d[e\omega, e\tau]^\gamma} e^{cd[\omega,\tau]^\gamma} e^{\phi(e\tau)} \\
&\leq e^{\alpha d[\omega,\tau]^\gamma} \left(L_\phi g\right)(\tau)
\end{aligned}$$

where $\omega, \tau \in E^{\mathbb{N}}$ with $i(\omega) = i(\tau)$. Hence for all $\omega, \tau \in E^{\mathbb{N}}$ with $i(\omega) = i(\tau)$ we have that $\left(T_\phi g\right)(\omega) \leq e^{\alpha d[\omega,\tau]^\gamma} \left(T_\phi g\right)(\tau)$. Also since $||T_\phi g||_\infty = 1$ we have that $\beta = e^\alpha \leq \left(T_\phi g\right)(\omega) \leq 1$ for all $\omega \in E^{\mathbb{N}}$. Thus T_ϕ

maps B into B and the Schauder fixed point theorem gives us that there exists $h \in B$ such that $T_\phi h = h$. Putting this another way we have $L_\phi h = \lambda h$, where $\lambda = ||L_\phi h||_\infty$ and since $h \in B$ we have that h is positive and $\lambda > 0$.

(2) We start by defining $\mathcal{N} := \{\hat{\rho} \mid \operatorname{supp}\hat{\rho} \subseteq E^{\mathbf{N}} \text{ and } \int h d\hat{\rho} = 1\}$, where h is as in (1). The Riesz representation theorem tells us that $\mathcal{N}$ can be regarded as a subspace of $C\left(E^{\mathbf{N}}\right)^*$, the space of bounded linear functionals on $C\left(E^{\mathbf{N}}\right)$, and it is easy to verify that $\mathcal{N}$ is convex and compact with respect to the weak* topology. Now let L_ϕ^* denote the dual mapping of L_ϕ defined on $C\left(E^{\mathbf{N}}\right)^*$ by

$$\int gd\left(L_\phi^*\hat{\rho}\right) = \int \left(L_\phi g\right) d\hat{\rho}$$

for all $g \in C\left(E^{\mathbf{N}}\right)$. Then given $\hat{\rho} \in \mathcal{N}$ we have

$$\int hd\left(\frac{1}{\lambda}L_\phi^*\hat{\rho}\right) = \int \frac{1}{\lambda}\left(L_\phi h\right) d\hat{\rho} = \int hd\hat{\rho} = 1.$$

Hence $\frac{1}{\lambda}L_\phi^*$ maps $\mathcal{N}$ into itself and the Schauder fixed point theorem gives us a measure $\hat{\mu} \in \mathcal{N}$ such that $\frac{1}{\lambda}L_\phi^*\hat{\mu} = \hat{\mu}$, which we can normalise to obtain $\hat{\mu}\left(E^{\mathbf{N}}\right) = 1$. Finally for all $g \in C\left(E^{\mathbf{N}}\right)$ we have

$$\lambda \int gd\hat{\mu} = \lambda \int gd\left(\frac{1}{\lambda}L_\phi^*\hat{\mu}\right) = \int \left(L_\phi g\right) d\hat{\mu}.$$

(3) Now let $\hat{\nu}$ be as defined in Equation 9 and let $g \in C\left(E^{\mathbf{N}}\right)$, then Lemma 3.5 gives us

$$\begin{aligned}
\int g(\omega)\, d\hat{\nu}(\omega) &= \int g(\omega)\, \overline{h}(\omega)\, d\hat{\mu}(\omega) \\
&= \lambda^{-1} \int g(\omega) \left(L_\phi \overline{h}\right)(\omega)\, d\hat{\mu}(\omega) \\
&= \lambda^{-1} \int \left(L_\phi\left((g \circ \sigma) \times \overline{h}\right)\right)(\omega)\, d\hat{\mu}(\omega) \\
&= \lambda^{-1}\lambda \int (g \circ \sigma)(\omega)\, \overline{h}(\omega)\, d\hat{\mu}(\omega) \\
&= \int g(\sigma(\omega))\, d\hat{\nu}(\omega).
\end{aligned}$$

■

Finally we are able to state and prove the theorem that gives the existence of Gibbs states.

Theorem 3.7 *Let $\lambda, \hat{\mu}$ and $\hat{\nu}$ be defined as in Theorem 3.6. Then we have $\log\lambda = P(\phi)$, where $P(\phi)$ is defined as in Equation 7. In particular this limit exists, is independent of the ω_τ that we choose and coincides with $\log\lambda$. We also have that there exists $a_1 > 0$ such that for all $n \in \mathbf{N}$ and $\omega \in E^{\mathbf{N}}$,*

$$a_1^{-1} \leq \frac{\hat{\mu}\left([\omega|n]\right)}{\exp\left(-nP(\phi) + S_n\phi(\omega)\right)}\,,\ \frac{\hat{\nu}\left([\omega|n]\right)}{\exp\left(-nP(\phi) + S_n\phi(\omega)\right)} \leq a_1. \tag{10}$$

Thus $\hat{\nu}$ is a Gibbs State. Further, the measure $\hat{\mu}$ satisfies

$$\hat{\mu}\left(\sigma^n(A)\right) = \exp\left(nP(\phi)\right) \int_A \exp\left(-S_n\phi(\omega)\right) d\hat{\mu}(\omega), \tag{11}$$

for every Borel set $A \subseteq E^{\mathbf{N}}$ and $n \in \mathbf{N}$. Moreover, any measure $\hat{\mu}$ satisfying

$$a_1^{-1} \leq \frac{\hat{\mu}\left([\omega|n]\right)}{\exp\left(-nP(\phi) + S_n\phi(\omega)\right)} \leq a_1$$

for all $n \in \mathbf{N}$ and $\omega \in E^{\mathbf{N}}$ is ergodic for σ.

Proof: Let us denote the indicator function of a set A by 1_A. For $\omega \in E^{\mathbf{N}}$ and $\tau \in E^{(*)}$ such that $t(\tau) = i(\omega)$ if $A \subseteq [\tau]$ then we have

$$\begin{aligned} L_\phi^{|\tau|}\left(e^{-S_{|\tau|}\phi(\omega)}1_A(\omega)\right) &= \exp\left(-S_{|\tau|}\phi(\tau\omega)\right)1_A(\tau\omega)\exp\left(S_{|\tau|}\phi(\tau\omega)\right) \\ &= 1_{\sigma^{|\tau|}(A)}(\omega), \end{aligned}$$

since $\omega \in \sigma^{|\tau|}(A)$ if and only if $\tau\omega \in A$. Thus integrating and using Equation 8 $|\tau|$ times we have

$$\begin{aligned} \hat{\mu}\left(\sigma^{|\tau|}(A)\right) &= \int 1_{\sigma^{|\tau|}(A)}(\omega)\,d\hat{\mu}(\omega) \\ &= \int L_\phi^{|\tau|}\left(e^{-S_{|\tau|}\phi(\omega)}1_A(\omega)\right)d\hat{\mu}(\omega) \\ &= \lambda^{|\tau|}\int e^{-S_{|\tau|}\phi(\omega)}1_A(\omega)\,d\hat{\mu}(\omega) \\ &= \lambda^{|\tau|}\int_A e^{-S_{|\tau|}\phi(\omega)}d\hat{\mu}(\omega). \end{aligned}$$

Equation 11 will follow if $\log\lambda = P(\phi)$. To verify this is the case let us set $A = [\tau]$, then using Equation 5 we have

$$e^{-b} \leq \lambda^{|\tau|}e^{-S_{|\tau|}\phi(\omega)}\hat{\mu}([\tau]) \leq e^b$$

for any $\omega \in [\tau]$. This in turn implies that

$$e^{-b}e^{S_{|\tau|}\phi(\omega)} \leq \lambda^{|\tau|}\hat{\mu}([\tau]) \leq e^b e^{S_{|\tau|}\phi(\omega)}$$

for any $\omega \in [\tau]$. Now summing over all τ of length n we get

$$e^{-b}\sum_{\tau\in E^{(n)}} e^{S_n\phi(\omega_\tau)} \leq \lambda^n \leq e^b\sum_{\tau\in E^{(n)}} e^{S_n\phi(\omega_\tau)},$$

where for each $\tau \in E^{(n)}$, $\omega_\tau \in [\tau]$. Rearranging, taking logarithms and dividing through by n gives that

$$\log\lambda - \frac{b}{n} \leq \frac{1}{n}\log\sum_{\tau\in E^{(n)}}\exp(S_n\phi(\omega)) \leq \log\lambda + \frac{b}{n}$$

for any $\omega \in \tau$. Thus we have that $\log\lambda = P(\phi)$. This also gives us that Equation 10 follows for $\hat{\mu}$. To verify that Equation 10 holds for $\hat{\nu}$ we observe that Equation 9 implies that

$$\hat{\mu}([\tau])\left(\inf_{g\in C(E^{\mathbf{N}})} g(\omega)\right) \leq \hat{\nu}([\tau]) \leq \hat{\mu}([\tau])\left(\sup_{g\in C(E^{\mathbf{N}})} g(\omega)\right)$$

where $0 < \inf_{g\in C(E^{\mathbf{N}})} g(\omega) \leq \sup_{g\in C(E^{\mathbf{N}})} g(\omega) < \infty$.

We now show that any measure $\hat{\rho}$ satisfying

$$a_1^{-1} \leq \frac{\hat{\rho}([\omega|n])}{\exp(-nP(\phi) + S_n\phi(\omega))} \leq a_1$$

for all $n \in \mathbf{N}$ and $\omega \in E^{\mathbf{N}}$ is ergodic for σ. To do this we show that $\hat{\mu}$ is ergodic for σ and deduce the result from the fact that all other measures satisfying the condition are equivalent to $\hat{\mu}$.

Let A be a σ-invariant subset of $E^{\mathbf{N}}$. For any $\tau \in E^{(*)}$ we have

$$\hat{\mu}(A) = \hat{\mu}\left(\sigma^{|\tau|}(A\cap[\tau])\right) = \exp|\tau|P(\phi)\int_{A\cap[\tau]}\exp\left(-S_{|\tau|}\phi(\tau)\right)d\hat{\mu}(\omega).$$

Thus Equation 5 implies that for all $\tau \in E^{(*)}$,

$$e^{-b}\hat{\mu}(A) \leq \exp\left(|\tau|P(\phi) - S_{|\tau|}\phi(\tau)\right)\hat{\mu}(A\cap[\tau]) \leq e^b\hat{\mu}(A).$$

Now since $E^{\mathbf{N}}$ is an invariant set, a similar argument gives that for all $\tau \in E^{(*)}$,

$$e^{-b} \leq \exp\left(|\tau| P(\phi) - S_{|\tau|}\phi(\tau)\right) \hat{\mu}([\tau]) \leq e^{b}.$$

Thus, combining these two equations, we get that

$$\hat{\mu}(A)\,\hat{\mu}([\tau]) \leq e^{2b}\hat{\mu}(A \cap [\tau]),$$

for all $\tau \in E^{(*)}$. The cylinders $\left\{[\tau] \middle| \tau \in E^{(*)}\right\}$ generate the Borel sets of $E^{\mathbf{N}}$ so we have

$$\hat{\mu}(A)\,\hat{\mu}(B) \leq e^{2b}\hat{\mu}(A \cap B),$$

for all Borel sets B.

Now taking $B = E^{\mathbf{N}}\backslash A$ gives $\hat{\mu}(A)\,\hat{\mu}\left(E^{\mathbf{N}}\backslash A\right) \leq e^{2b}\hat{\mu}\left(A \cap \left(E^{\mathbf{N}}\backslash A\right)\right) = 0$ so either $\hat{\mu}(A) = 0$ or $\hat{\mu}\left(E^{\mathbf{N}}\backslash A\right) = 0$ as required. ∎

Finally we show that Gibbs states are unique.

Theorem 3.8 *Let $\hat{\mu}$ and $\hat{\nu}$ be two Gibbs states on $E^{\mathbf{N}}$. Then $\hat{\mu} = \hat{\nu}$.*

Proof: Since both $\hat{\mu}$ and $\hat{\nu}$ satisfy Equation 6 for some a_0, we can deduce that there exists $c \in (0, \infty)$ such that for all $\tau \in E^{(*)}$, $\hat{\mu}([\tau]) \leq c\hat{\nu}([\tau])$. Also, since the Borel sets of $E^{\mathbf{N}}$ are generated by these cylinders, $\hat{\mu}(B) \leq c\hat{\nu}(B)$ for all $B \in \mathcal{B}\left(E^{\mathbf{N}}\right)$. In particular, if $\hat{\nu}(E) = 0$ then $\hat{\mu}(E) = 0$, thus $\hat{\mu} \ll \hat{\nu}$. The result now follows from the following well known proposition. ∎

Proposition 3.9 *Let μ and ν be T-ergodic probability measures on X such that $\mu \ll \nu$. Then $\mu = \nu$.*

3.4 Entropy and the Variational Principle

Let $\hat{\mu}$ be a σ-invariant measure on $E^{\mathbf{N}}$, the entropy of the dynamical system $\left(E^{\mathbf{N}}, \sigma\right)$ with respect to $\hat{\mu}$ quantifies the rate at which information can be obtained from the system by repeated observations. Heuristically we perform the following experiment. We take a list of vertices $v_1, \ldots, v_n$ and ask how many $\omega \in E^{\mathbf{N}}$ have this list of vertices as the initial vertices of their first n iterates under σ *i.e.* what is the $\hat{\mu}$ measure of $A = \left\{\omega \in E^{\mathbf{N}} \mid i(\omega) = v_1, \ldots, i\left(\sigma^{n-1}(\omega)\right) = v_n\right\}$? Another way to look at this is, given an $\omega \in E^{\mathbf{N}}$ how accurately can we determine the location of ω by knowing what the initial vertex of its first n iterates are. If the measure of A is small then the location of ω is well determined by the observations, if large then not so well determined.

Now let us suppose that the measure of A scales like c^n, so that with each iterate we increase our knowledge by a factor of approximately c. The entropy of the system with respect to the measure $\hat{\mu}$ is the limit as $n \to \infty$ of the average amount of information gained from admissible sequences of n vertices. Thus if we note that $\hat{\mu}(A) = \hat{\mu}([\omega|n])$ then we have the following definition. The *entropy of σ with respect to the measure $\hat{\mu}$* is given by,

$$h_{\hat{\mu}}(\sigma) = \lim_{n \to \infty} -\frac{1}{n} \sum_{\tau \in E^{(n)}} \hat{\mu}([\tau]) \log \hat{\mu}([\tau]). \tag{12}$$

We now prove that this limit exists.

Lemma 3.10 *Let $\hat{\mu}$ be a σ-invariant probability measure supported on $E^{\mathbf{N}}$. Then the entropy $h_{\hat{\mu}}(\sigma)$, defined in Equation 12, exists.*

Proof: We prove this by showing that the sequence $\left(\sum_{\tau \in E^{(n)}} \hat{\mu}([\tau]) \log \hat{\mu}([\tau])\right)_n$ is subadditive.

Let us start by defining $\Phi\colon [0, \infty) \to \mathbf{R}$ by setting

$$\Phi(x) = \begin{cases} -x \log x & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then Φ is concave. Now if $\hat{\mu}\left(E_u^{\mathbf{N}}\right) > 0$ for all $u \in V$ then given $m, n \in \mathbf{N}$ and $\tau \in E^{(n)}$ we have

$$\begin{aligned}
\Phi\left(\hat{\mu}\left([\tau]\right)\right) &= \Phi\left(\sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\tau\alpha]\right)\right) \\
&= \Phi\left(\sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\alpha]\right) \hat{\mu}\left([\tau\alpha]\right) / \hat{\mu}\left([\alpha]\right)\right) \\
&\geq \sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\alpha]\right) \Phi\left(\hat{\mu}\left([\tau\alpha]\right) / \hat{\mu}\left([\alpha]\right)\right) \\
&\geq \sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\alpha]\right) \left(\hat{\mu}\left([\tau\alpha]\right) / \hat{\mu}\left([\alpha]\right)\right) \left(\log \hat{\mu}\left([\alpha]\right) - \log \hat{\mu}\left([\tau\alpha]\right)\right) \\
&\geq \sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\tau\alpha]\right) \log \hat{\mu}\left([\alpha]\right) + \sum_{\alpha \in E^m_{i(\tau)}} \Phi\left(\hat{\mu}\left([\tau\alpha]\right)\right)
\end{aligned}$$

using the definition of Φ and the fact that it is concave. Now by neglecting any α such that $\hat{\mu}\left([\alpha]\right) = 0$ we can obtain this inequality without the restriction that $\hat{\mu}\left(E_u^{\mathbf{N}}\right) > 0$ for all $u \in V$. Thus

$$\sum_{\tau \in E^{(n)}} \Phi\left(\hat{\mu}\left([\tau]\right)\right) \geq \sum_{\alpha \in E^m_{i(\tau)}} \hat{\mu}\left([\alpha]\right) \log \hat{\mu}\left([\alpha]\right) + \sum_{\tau\alpha \in E^{n+m}} \Phi\left(\hat{\mu}\left([\tau\alpha]\right)\right).$$

Putting this another way, we have

$$\sum_{\tau \in E^{n+m}} \Phi\left(\hat{\mu}\left([\tau]\right)\right) \leq \sum_{\tau \in E^n} \Phi\left(\hat{\mu}\left([\tau]\right)\right) + \sum_{\tau \in E^m} \Phi\left(\hat{\mu}\left([\tau]\right)\right)$$

for all $m, n \in \mathbf{N}$. ■

Note: The entropy $h_{\hat{\mu}}(\sigma)$ is usually defined in terms of more general partitions than the cylinders we have used but the limits are equal in this setting.

Our reason for defining entropy is that it is related to topological pressure via a formula, usually known as the the variational principle. We will now go on to look at the variational principle but before we can do this we require the following theorem.

Theorem 3.11 *Let $\hat{\mu}$ be a σ-invariant probability measure on $E^{\mathbf{N}}$ and $\phi\colon E^{\mathbf{N}} \to \mathbf{R}$ be a γ-Hölder continuous function. Then*

$$\int \phi(\omega)\, d\hat{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum_{\tau \in E^{(n)}} S_n\phi(\tau)\, \hat{\mu}\left([\tau]\right). \tag{13}$$

Proof: First since $\hat{\mu}$ is σ-invariant we have that for all $i \in \mathbf{N}$, $\int \phi(\omega)\, d\hat{\mu} = \int \phi\left(\sigma^i(\omega)\right) d\hat{\mu}$. Thus we can deduce that $\int \phi(\omega)\, d\hat{\mu} = \frac{1}{n} \int \sum_{i=0}^{n-1} \phi\left(\sigma^i(\omega)\right) d\hat{\mu} = \int S_n\phi(\omega)\, d\hat{\mu}$. Now using this we have

$$\begin{aligned}
\left| \int \phi(\omega)\, d\hat{\mu} - \frac{1}{n} \sum_{\tau \in E^{(n)}} S_n\phi(\tau)\, \hat{\mu}\left([\tau]\right) \right| &= \left| \frac{1}{n} \sum_{\tau \in E^{(n)}} \left(\int_{\omega \in [\tau]} S_n\phi(\omega)\, d\hat{\mu} - S_n\phi(\tau)\, \hat{\mu}\left([\tau]\right) \right) \right| \\
&\leq \frac{1}{n} \sum_{\tau \in E^{(n)}} \hat{\mu}\left([\tau]\right) \sup_{\omega \in [\tau]} \left| S_n\phi(\omega) - S_n\phi(\tau) \right| \\
&\leq b/n.
\end{aligned}$$

Letting $n \to \infty$ gives the result. ■

Finally we state and prove the *Variational Principle.*

Theorem 3.12 *Let $\phi \colon E^{\mathbf{N}} \to \mathbf{R}$ be a γ-Hölder continuous function. Then,*

$$P(\phi) = \sup \left\{ h_{\hat{\mu}} + \int \phi d\hat{\mu} \;\middle|\; \hat{\mu} \text{ is a } \sigma\text{-invariant probability measure on } E^{\mathbf{N}} \right\}. \tag{14}$$

Moreover, this supremum is obtained by the Gibbs state i.e the measure $\hat{\nu}$ defined in Equation 9.

Proof: For $n = 0, 1, 2, \ldots$ and $\hat{\mu}$ a σ-invariant probability measure on $E^{\mathbf{N}}$, define

$$s_n(\hat{\mu}) = \frac{1}{n} \sum_{\tau \in E^{(n)}} \hat{\mu}([\tau]) \left(-\log \hat{\mu}([\tau]) + S_n\phi(\tau)\right).$$

Now Lemma 3.2 tells us that

$$s_n(\hat{\mu}) \leq \frac{1}{n} \log \left(\sum_{\tau \in E^{(n)}} \exp\left(S_n\phi(\tau)\right) \right).$$

Letting $n \to \infty$ and using the definitions of entropy and pressure and Equation 13 gives

$$h_{\hat{\mu}}(\sigma) + \int \phi d\hat{\mu} \leq P(\phi).$$

To complete the proof we require to show that $h_{\hat{\nu}}(\sigma) + \int \phi d\hat{\nu} \geq P(\phi)$. We observe that,

$$\begin{aligned}
s_n(\hat{\nu}) &\geq \frac{1}{n} \sum_{\tau \in E^{(n)}} \hat{\nu}([\tau]) \left(-\log\left(a_1 \exp\left(-nP(\phi) + S_n\phi(\tau)\right)\right) + S_n\phi(\tau)\right) \\
&= \frac{1}{n} \sum_{\tau \in E^{(n)}} \hat{\nu}([\tau]) \left(a_1 + nP(\phi)\right) \\
&= P(\phi) - \frac{1}{n} \log a_1,
\end{aligned}$$

where a_1 is the constant appearing in Theorem 3.7. Now letting $n \to \infty$ gives the result. ∎

A measure that obtains the supremum in Equation 14 is called an *equilibrium* measure. Thus a Gibbs state is an equilibrium measure.

4 Multifractal Analysis

4.1 An Introduction to Multifractals

Many physical objects are distributed with widely varying intensity. For example, if we consider the distribution of the rate of dissipation of energy in a turbulent fluid flow then we discover that the rate of dissipation is very different at different points in the flow. A second example is the time frequency distribution of orbits on the attractors of chaotic dynamical systems. Another example comes from considering the probability of a random-walk reaching a certain point on a diffusion limited aggregation (DLA). These are just a few of many situations in the physical sciences where we find this type of behaviour. For many other examples we refer the reader to [Fed88]. These phenomenon are examples of naturally occurring measures. For example, we define the occupation measure of an attractor of a chaotic dynamical system using the following intuitive rule: let the measure of a subset of the attractor be proportional to the amount of time which a typical orbit will spend in that subset of the attractor. The measures listed above all have one property in common. If we consider the set of αs for which there are a large number of points x such that the measure of balls with centre x and radius r scales like r^{α} then we find that this set of αs is large. Any measure with this property is called a multifractal measure. The aim of multifractal analysis is to find useful methods to study and characterise multifractal measures.

Modern multifractal analysis was born from two independent circles of people considering how to characterise these types of measures or distributions. The first circle's work can be traced back to two early papers by Mandelbrot, [Man72] and [Man74]. In these papers Mandelbrot suggested that the distribution of intermittent dissipation of energy in highly turbulent fluid flows is multifractal in nature and studied it by calculating its moments. Mandelbrot's ideas were taken up by Frisch and Parisi in [FP85]. In a now famous appendix to this paper they gave an intuitive interpretation of what their analysis involved. In [BPPV84], Benzi et al. also considered Mandelbrot's ideas and extended them to include the occupation measures of attractors of chaotic dynamical systems (often called strange sets).

Independently, in [Gr83], [GP83] and [HP83] Grassberger, Hentschel and Procaccia proposed characterising occupation measures using extensions of ideas introduced by Rènyi in the sixties (see [Re57], [Re60] and [Re61]). At that time the main method for characterising occupation measures was to calculate three dimensions, the box-counting dimension, the information dimension and the correlation dimension. In the sixties Rènyi had introduced a countably infinite set, $(D_q)_{q\in\mathbf{Z}}$, of numbers which coincided with the box-counting, information and correlation dimensions for $q = 0, 1$ and 2 respectively. The basic idea of Grassberger, Hentschel and Procaccia was to extend this family of numbers to non-integer values. An occupation measure could then be characterised according to this family of numbers, $(D_q)_{q\in\mathbf{R}}$. As we will see later all this really involved doing was calculating the moments of the occupation measure.

In the mid-eighties these two circles merged and a significant breakthrough occurred. In [HJKPS86], Halsey et al. introduced a general multifractal formalism which included the celebrated $f(\alpha)$ function often called the *spectrum of singularities* or *multifractal spectrum*.

We now introduce the multifractal spectrum. The key idea in multifractal analysis is to consider the size of the set of points x where the measure of a ball of radius r scales like r^{α}. We thus decompose the support of a measure into sets which have the same scaling behaviour and calculate the dimension of the sets in this decomposition. Formally we make the following definitions.

Let $\mu \in \mathcal{M}^1(X)$ and $x \in \operatorname{supp}\mu$, then the *lower and upper local dimensions* of μ at x are defined to be:

$$\underline{\alpha}_{\mu}(x) = \liminf_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r}$$

and

$$\overline{\alpha}_{\mu}(x) = \limsup_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively. If $\underline{\alpha}_{\mu}(x) = \overline{\alpha}_{\mu}(x)$, then we refer to the common value as the *local dimension* of μ at x and denote it by $\alpha_{\mu}(x)$. For $\alpha \geq 0$, we define the *Hausdorff and packing multifractal spectra* of μ by

$$f_{\mu}(\alpha) = \dim_{\mathrm{H}} \{x \in \operatorname{supp}\mu \mid \alpha_{\mu}(x) = \alpha\}$$

and

$$F_\mu(\alpha) = \dim_P \{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\}$$

respectively, where $\dim_H$ and $\dim_P$ denote Hausdorff and packing dimension respectively.

We note that these two spectra are what are called fine grain spectra. Mathematicians usually study fine grain spectra whereas physicists are generally more interested in the coarse grain multifractal spectrum of a measure, described below. One of the major achievements of early mathematical work in the field of multifractal analysis was to show that these two types of spectra coincide for a large class of measures.

We now consider the results of Halsey et al. found in [HJKPS86]. In this paper they define and show the relationship between three important functions, the coarse grain multifractal spectrum, the generalised Rènyi dimensions and an auxiliary function τ. Let μ be a Borel probability measure on $\mathbf{R}^d$, $l \in \mathbf{R}$ and $(E_i)_i$ be a partition of the support of μ such that $l_i = \operatorname{diam} E_i < l$. Also, set $p_i = \mu(E_i)$. Halsey et al. start by observing that in different regions of the support of the measure different scaling behaviours can exist, that is there exists a range of α such that p_i scales like l_i^α. They then claim that if the partition of the support of μ is made by sets of the same size *i.e.* we set $l_i = l$ for each i, then the number of times α takes on a value between α' and $\alpha' + d\alpha'$ is of the form

$$d\alpha' \rho(\alpha') l^{-f(\alpha')},$$

where f is a continuous function and ρ is a weight function. It is this function f which we call the *coarse grain multifractal spectrum of* μ. In other words Halsey et al. give a definition of the coarse grain multifractal spectrum which can be formalised in the following way: first, given $\alpha \in \mathbf{R}$, $\epsilon > 0$ and $n \in \mathbf{N}$, set

$$A(\alpha, \epsilon, n) = \left\{ x \in \operatorname{supp}\mu \middle| \alpha \le \frac{\log \mu(B(x,r))}{\log r} \le \alpha + \epsilon, \text{ for } r < \frac{1}{n} \right\}.$$

Also, let $N(\alpha, \epsilon, n)$ be the smallest number of balls of radius less than $\frac{1}{n}$ that can be used to cover $A(\alpha, \epsilon, n)$. Further, set

$$S(\alpha, \epsilon) = \lim_{n\to\infty} \frac{\log N(\alpha, \epsilon, n)}{n},$$

and

$$f(\alpha) = \lim_{\epsilon \searrow 0} S(\alpha, \epsilon).$$

Thus in Halsey et al.'s words the coarse grain singularity spectrum

> reflects the differing dimensions of the sets upon which the singularities of strength α' may lie ... Thus, we model fractal measures by interwoven sets of singularities of strength α, each characterised by its own dimension.

Next Halsey et al. recall the definition of D_q, the generalised Rènyi dimension,

$$D_q = \lim_{l \searrow 0} \left[\frac{1}{q-1} \frac{\log \chi(q)}{\log l} \right],$$

where $\chi(q) = \sum_i p_i^q$. We note that they do not give a definition for $q = 1$. Finally they define a function $\tau \colon \mathbf{R} \to \mathbf{R}$ by setting $\tau(q)$ equal to the unique number such that,

$$\lim_{l \searrow 0} \sum_i \frac{p_i^q}{l_i^\tau} = \begin{cases} \infty & \text{for } \tau > \tau(q) \\ 0 & \text{for } \tau < \tau(q). \end{cases} \tag{15}$$

It is natural to question whether such functions exist and in fact much work has been done on proving that these functions (or analogies of these functions) do exist for certain measures e.g. [Ra89], [CM92] and [EM92]. In the next section we will discuss one example of this work *i.e.* Rand's paper [Ra89]. With these definitions behind us we can now summarise Halsey et al.'s main results concerning the relationship between these three functions. First, D_q is related to the τ function by

$$\tau(q) = (q-1)\,D_q.$$

More interestingly, Halsey et al. define $\alpha(q)$ using the extremal condition

$$\frac{d}{d\alpha'}\left[q\alpha' - f(\alpha')\right]|_{\alpha'=\alpha(q)} = 0$$

and using this definition they find that

$$D_q = \frac{1}{q-1}\left[q\alpha(q) - f(\alpha(q))\right].$$

Halsey et al.'s results form part of a more general body of results in the Physics literature which we summarise in Folklore Theorem 4.1 usually known as the *multifractal formalism.* This folklore theorem is arranged so that the parts are ordered according to historical development. Part (1) tells us how to calculate the τ function, part (2) tells us what properties the τ function has and part (3) tells us how the τ function is related to the multifractal spectrum. First, let $f\colon \mathbf{R}\to\mathbf{R}$ be a real-valued function, the *Legendre transform* of f is the function $f^*\colon \mathbf{R}\to[-\infty,\infty)$ defined by

$$f^*(x) = \inf_y\,(xy + f(y))$$

for $x\in\mathbf{R}$. We observe that the Legendre transform of a function f is concave and that if f is differentiable and strictly convex and decreasing then a simple argument from calculus gives that,

$$f^*(-f'(q)) = -qf'(q) + f(q).$$

Folklore Theorem 4.1 *Let τ be the function defined in Equation 15. Then the following hold:*

1. *The function τ can be calculated using the following box-counting argument,*

$$\tau(q) = \lim_{n\to\infty}\frac{\log\left(\sum_{C\in\mathcal{C}_n,\mu(C)>0}\mu(C)^q\right)}{-\log 2^{-n}},$$

   where for $n\in\mathbf{N}$,

$$\mathcal{C}_n = \left\{\prod_{i=1}^{d}\left[\frac{k_i}{2^n},\frac{k_i+1}{2^n}\right)\,\middle|\,k_i\in\mathbf{Z}\right\}.$$

2. (a) *τ is decreasing, convex and smooth.*
   (b) *τ has affine asymptotes as $q\to\pm\infty$.*
   (c) *$\tau(1)=0$.*
   (d) *The line with slope 1 passing through the origin is a tangent to the graph of τ^*, the Legendre transform of τ.*

3. *There exist numbers $0\le\underline{a}\le\overline{a}\le\infty$ such that,*

$$f_\mu(\alpha) = \begin{cases}\tau^*(\alpha) & \alpha\in[\underline{a},\overline{a}]\\ 0 & \alpha\notin[\underline{a},\overline{a}].\end{cases}$$

When we first heard of the proposed link between τ and f we were tempted to ask, 'given τ why would you consider f?' In order to understand why we must turn to statistical mechanics and discuss its links with multifractal analysis. Let us consider a physical system which can take the following set of finite or countably infinite states $1,2,\ldots$ with energies $E_1,E_2,\ldots$. Also, let us suppose that the probability that the system is in state i with energy E_i is given by p_i. Given such a system a physicist would consider the following three quantities. The *average energy* of the system

$$\mathcal{U} = \sum_i p_i E_i,$$

the *entropy* of the system

$$\mathcal{S} = -\sum_i p_i \log p_i$$

and the *free energy* of the system

$$\mathcal{F} = \mathcal{U} - T\mathcal{S},$$

where T is the temperature of the system.

A normalised version of Boltzman's law tells us that p_i is proportional to $e^{-\frac{1}{T}E_i}$, thus writing $\beta = \frac{1}{T}$ we obtain

$$p_i = \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}} = \frac{e^{-\beta E_i}}{Z},$$

where $Z = \sum_i e^{-\beta E_i}$ is called the *partition function* of the system. This partition function turns out to be very important. First, differentiation of Z with respect to β gives,

$$Z' = \sum_i -E_i e^{-\beta E_i} = \sum_i -E_i \left(p_i Z\right) = -Z \sum_i p_i E_i.$$

Thus,

$$\mathcal{U} = -\frac{Z'}{Z} = -\left(\log Z\right)'.$$

Second, the properties of $\log Z$ give that

$$\mathcal{S} = -\sum_i p_i \log p_i = -\sum_i p_i \left(-\beta E_i - \log Z\right) = \beta \sum_i p_i E_i + \log Z = \beta\, \mathcal{U} + \log Z = -\beta \left(\log Z\right)' + \log Z$$
$$= \left(\log Z\right)^*.$$

Finally, substitution gives that

$$\mathcal{F} = -\frac{1}{\beta} \log Z.$$

Using these results from statistical mechanics we now wish to motivate considering $f = \tau^*$ given τ. The main idea is simple; the τ function can be viewed as a partition function and thus it is natural to consider its Legendre transform because this is like considering the 'entropy' of the system. To see this more clearly we note that part (1) of Folklore Theorem 4.1 tells us that in calculating τ physicists consider partitions using sets of the same size. Now if we consider Equation 15 with l_i equal to some constant for each i, we obtain the following

$$\tau\left(q\right) \sim \log \sum_i p_i^q = \log \sum_i e^{-q \log p_i^{-1}}.$$

Thus with $E_i = \frac{1}{p_i}$ and $\beta = q$, we have that τ can be interpreted as a partition function.

The first attempts at developing rigourous analogies of the above proposition can be found in the work of Collet et al. [CLP87] and Rand [Ra89]. These papers both developed the above box-counting type arguments in the context of specific measures. Following on from these papers much rigourous work has been done on calculating the multifractal spectra of specific measures. The following is a list of the important types of measure which have been analysed together with appropriate references.

1. Self-similar measures.
    (a) Self-similar measures ([BMP92], [CM92] and [LN1]).
    (b) Graph directed self-similar measures ([EM92] and [Ol95]).
    (c) Self-similar measures generated by infinite IFSs ([MR95]).
    (d) Random self-similar measures ([Ol94], [Fa94] and [AP96]).
    (e) Random graph directed self-similar measures ([Ol94]).
    (f) Vector-valued self-similar measures ([FO96]).
2. Self-affine measures ([Ki92], [Ol98] and [SS1]).
3. Gibbs states
    (a) Gibbs states ([Ra89], [KG92] and [PW97]).
    (b) Random Gibbs states ([Kif95])
4. Measures of maximal entropy for hyperbolic rational maps ([Lo89]).
5. Harmonic measures on Julia sets ([CDM92]).

An important feature of the analyses listed is that they all fall into two types. They either use geometric arguments together with the ergodic theorem to calculate the spectrum or they use the thermodynamic formalism as developed by Bowen and Ruelle to calculate the spectrum. Of particular importance in this list is the paper by Cawley and Mauldin on self-similar measures which we will discuss in detail in the next section. This paper is important because by choosing to analyse self-similar measures Cawley and Mauldin found a setting in which they could perform their analysis by using standard techniques from calculus and geometric measure theory rather than by using the thermodynamic formalism. This opened up the field of multifractal analysis to many mathematicians who had previously been deterred by their lack of knowledge of the thermodynamic formalism.

In the early nineties, in an attempt to move away from the policy of concentrating on specific measures, several authors chose to develop more general theory which covered the analysis of wide classes of measures. The most important work of this type is that of Peyrière [BMP92] and [Pey92], Pesin [Pes88], [Pes91] and [Pes93] and Olsen [Ol95]. In the third section of this chapter we will develop Olsen's multifractal formalism. Olsen's formalism covers Borel probability measures on metric spaces. It is based on the heuristics of Halsey et al. and introduces rigourous multifractal generalisations of the centred Hausdorff and packing measures and the Hausdorff and packing dimensions. In [Ol95] Olsen studies these generalised measures and dimensions and shows how they are related to the multifractal spectra of the measures being analysed. Included in his paper are general theorems which allow one to calculate the spectrum of a measure given its dimension function provided the measure is sufficiently regular (see Theorem 4.19). In later papers, [Ol96] and [Ol1], Olsen has extended this general theory to consider product measures and slice measures and used it to calculate the spectra of random graph directed self-similar measures.

In summary, from a mathematical perspective we feel that four major steps have been taken in the field of multifractal analysis. First, the idea of analysing measures with widely varying intensity by calculating their moments, inspired the field itself. Second, early attempts were made to provide rigourous analogies of the intuitive ideas in the physics literature by using the thermodynamic formalism in the setting of specific measures cf. [Ra89] and [CLP87]. Third, Cawley and Mauldin opened up the field to geometric measure theorists by analysing multifractal measures without specifically mentioning the thermodynamic formalism. Finally, mathematicians began to introduce general formalisms covering wide ranges of measures rather than specific measures. The remainder of this chapter is spent looking at the final three stages in this development. In the next section we consider both an early attempt at bringing rigour to multifractal analysis and the move towards the use of ideas from geometric measure theory in multifractal analysis. We do this by discussing the results from two influential papers [Ra89] and[CM92]. Then in the third section of this chapter we move on to discuss Olsen's multifractal formalism.

4.2 Two Influential Mathematical Papers

The primary purpose of this section is to discuss two influential papers in the mathematical development of multifractal analysis. These two papers are Rand's 1989 paper, *The singularity spectrum $f(\alpha)$ for cookie-cutters* and Cawley and Mauldin's 1992 paper, *Multifractal Decompositions of Moran Fractals.* The discussion of both of these papers will include the main results of the paper and a brief discussion of how the author(s) proved these results. We note that the important results in these two papers *i.e.* those relating to the spectra of the measures, follow as corollaries to the results we obtain in Chapter 6.

We start by considering Rand's paper. Together with [CLP87] this paper marks the beginning of mathematical multifractal analysis. In this paper Rand considers Gibbs measures on cookie-cutters and proves rigourous analogies of the results found in Halsey et al. [HJKPS86].

A cookie-cutter system is a special type of dynamical system which is topologically conjugate to the code space $\{0,1\}^{\mathbf{N}}$ together with the natural shift map. Let $I = [0,1]$, choose x_0 and x_1 such that $0 < x_0 < x_1 < 1$ and set $I_0 = [0, x_0]$ and $I_1 = [x_1, 1]$. A cookie-cutter map is a map $g\colon I_0 \cup I_1 \to I$ such that

1. $g(I_0) = g(I_1) = I$.

2. g is γ-Hölder continuous for some $\gamma > 0$.

3. $|g'(x)| > 1$ for all $x \in I_0 \cup I_1$.

Thus a typical cookie-cutter map would have the following type of graph.

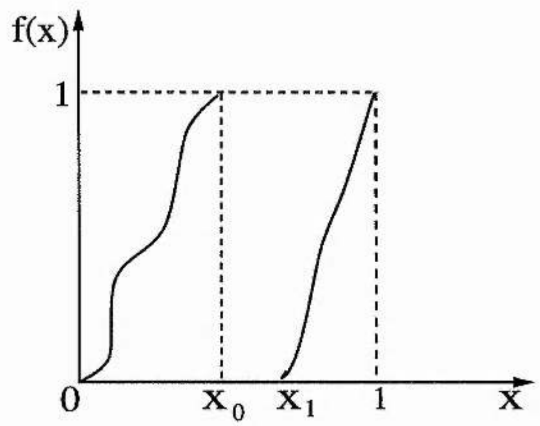


Figure 9: A cookie-cutter map

The cookie-cutter set $K(g)$ associated with g is

$$K = K(g) = \{x \in I_0 \cup I_1 \mid \forall n \in \mathbf{N},\ g^n(x) \in I_0 \cup I_1\}$$

For $n \in \mathbf{N}$ set $K(n) = \left\{x \in I_0 \cup I_1 \mid 1 \le j \le n \in \mathbf{N},\ g^j(x) \in I_0 \cup I_1\right\}$. The set $K(n)$ consists of 2^n closed intervals. We call these 2^n intervals the *geometric n-cylinders* and we denote the set of geometric n-cylinders by $\mathcal{C}_n$. Given $x \in K$ there exists a unique $C \in \mathcal{C}_n$ such that $x \in C$; we denote this C by $C_{x,n}$. Finally, these definitions allow us to define a natural map $\pi\colon E^{\mathbf{N}} = \{0,1\}^{\mathbf{N}} \to K$.

In the previous chapter we considered the thermodynamic formalism on the code space. We now wish to consider the thermodynamic formalism on cookie-cutters. Some consideration shows that all of the arguments used in that chapter can equally well be applied to this conjugate dynamical system provided that Hölder continuous functions on K satisfy the principle of bounded variation. In Lemma 1 of [Ra89] Rand proves this and thus we can deduce that given ϕ, a γ-Hölder continuous function on I, there exists a unique g-invariant probability measure μ_ϕ on K satisfying the following property: there exist constants $P = P(\phi)$ and $c \in (0, \infty)$ such that for all $n \in \mathbf{N}$, $C \in \mathcal{C}_n$ and $x \in C$,

$$c^{-1} e^{-nP + S_n\phi(x)} \leq \mu_\phi(C) \leq c\, e^{-nP + S_n\phi(x)},$$

where $S_n\phi(x) = \sum_{j=0}^{n-1} \phi\left(g^j(x)\right)$. The measure μ_ϕ is called the *Gibbs state* of ϕ and the constant $P(\phi)$ the *topological pressure* of ϕ. We can also deduce that

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \sum_{C\in\mathcal{C}_n} e^{S_n\phi(C)},$$

where $S_n\phi(C)$ can be chosen to take the value $S_n\phi(x)$ for any $x \in C$.

One further piece of information about Gibbs States which we require is that a Gibbs state ν of a Hölder continuous function ϕ_ν has the property that ν is non-singular in the following sense: if $\nu(g(A)) > 0$ then $\nu(A) > 0$ for all measurable sets A and the Radon-Nikodym derivative

$$J(x) = \lim_{A\to x} \frac{\nu(g(A))}{\nu(A)}$$

exists ν-almost surely and is equal to a Hölder continuous function, let us denote it by J (See Chapter 10 in [Par82]).

For the remainder of our discussion of Rand's paper we fix ν to be a Gibbs state on a cookie-cutter determined by g and let $\phi_1 = -\log|g'|$ and $\phi_2 = -\log J$, where J is the the Hölder continuous function above.

Having discussed the requisite preliminaries Rand follows the approach of Halsey et al. in introducing the singularity spectrum. In particular, he makes the following definitions: given A and L, two open intervals, let $N_n(A, L)$ denote the number of geometric n-cylinders C such that $l(C) := \frac{1}{n}\log \operatorname{diam}(C) \in L$ and $\alpha(C) := \log\nu(C) / \log \operatorname{diam}(C) \in A$. Also, let

$$S(A, L) = \liminf_{n\to\infty} \frac{\log N_n(A, L)}{n} \tag{16}$$

and

$$S(\alpha, l) = \inf\{S(A, L) \mid \alpha \in A,\ l \in L\}.$$

In Lemma 2 of [Ra89] Rand uses convex analysis to prove that $S(\alpha, l)$ is continuous and concave in each of its arguments and that

$$S(A, L) = \sup\{S(\alpha, l) \mid \alpha \in A,\ l \in L\}.$$

Finally, let

$$f(\alpha, l) = -S(\alpha, l)/l$$

and

$$f(\alpha) = \sup_l f(\alpha, l).$$

Rand (page 529) summarises this definition in the following way (adapted to our notation):

> Very roughly speaking, if $\mathcal{B}$ is a typical cover of K by non-over-lapping intervals of length ϵ then the number of $B \in \mathcal{B}$ with $\alpha(B) = \log\nu(B)/\log\epsilon \in [\alpha, \alpha + d\alpha]$ grows as $\epsilon \searrow 0$ as $\epsilon^{-f(\alpha)}$ or, in terms of cylinders, the number of geometric n-cylinders C such that $\nu(C) = (\operatorname{diam}(C))^\alpha$ grows, as $n \to \infty$, like $\epsilon^{-f(\alpha)}$ where $\epsilon = e^{nl}$ and $l = l(\alpha)$ is the value of l for which the supremum of $f(\alpha, l)$ is attained *i.e.* describes the dominant length scale.

This description makes it clear that the spectrum introduced by Rand is the coarse grain multifractal spectrum. Though Equation 16 makes it tempting to interpret f as the box-counting spectrum this interpretation is incorrect. The correct interpretation is of a statistical nature as demonstrated by Rand when considering the scaling behaviour of the number of $B \in \mathcal{B}$ that satisfy a certain property. This being clear we note that one of the major achievements of Rand's paper (Theorem 1 in [Ra89]) is to show that this coarse grain singularity spectrum coincides with the fine grain Hausdorff cylinder spectrum $f_c(\alpha) = \dim_H(\Sigma(\alpha))$ where $\Sigma(\alpha) = \{x \in K \mid \alpha(C_{n,x}) \to \alpha \text{ as } n \to \infty\}$. We also note that the fine grain Hausdorff cylinder spectrum coincides with the fine grain Hausdorff multifractal spectrum (see [CM92]).

Rand's next step is to use the thermodynamic formalism to introduce an auxiliary function τ. Later in the paper (Proposition 1 of [Ra89]) he shows that this auxiliary function τ has similar properties to the τ function introduced by Halsey et al. *i.e.* it satisfies

$$\lim_{n\to\infty} \sum_{C \in \mathcal{C}_n} \nu(C)^q \operatorname{diam}(C)^\tau = \begin{cases} \infty & \text{if } \tau < \tau(q) \\ 0 & \text{if } \tau > \tau(q). \end{cases}$$

Rand defines τ in the following way. Given $q, \tau \in \mathbf{R}$ let $\phi_{q,\tau} = \tau\phi_1 + q\phi_2$ and let $P(q,\tau) = P(\phi_{q,\tau})$. Then $P(q,\tau)$ is a concave real analytic function of q and τ and the implicit function theorem tells us that there exists a function $\tau(q)$ such that $P(q,\tau(q)) \equiv 0$. The details of this statement can be found in Section 4 of Rand's paper (pages 533 and 534) and use general results from the thermodynamic formalism and convex analysis.

Rand's next aim is to show that τ is related to the singularity spectrum f. Using geometric estimates, the principle of bounded variation, and general results from the thermodynamic formalism, he is able to prove the following important theorem (Theorem 2 in [Ra89]).

Theorem 4.2 *The singularity spectrum of f is real analytic and τ is the Legendre transform of f.*

Since for any convex function f we have that $(f^*)^* = f$ we can deduce from the fact that f is convex that $\tau^* = f$. Thus we find that the third part of the multifractal formalism holds for this particular type of measure.

Rand's final contribution in this paper is to consider the relationship between the generalised Rényi dimensions and the auxiliary function τ. Given $\mathcal{A}$, a δ-cover of $E \subseteq \mathbf{R}^d$, let

$$D_\mu^q(\mathcal{A}) = \begin{cases} (1-q)^{-1}\left(\log \sum_{A\in\mathcal{A}} \mu(A)^q\right)/\log\delta^{-1} & q \neq 1 \\ \left(\log \sum_{A\in\mathcal{A}} \mu(A)\log\mu(A)\right)/\log\delta^{-1} & q = 1. \end{cases}$$

For $q \leq 1$ set,

$$D_\mu^q(E,\delta) = \inf\left\{D_\mu^q(\mathcal{A}) \mid \mathcal{A} \text{ is a } \delta- \text{ cover of } E\right\}$$

and let $D_\mu^q(E) = \liminf_{\delta\searrow 0} D_\mu^q(E,\delta)$ and for $q > 1$ set

$$D_\mu^q(E,\delta) = \sup\left\{D_\mu^q(\mathcal{A}) \mid \mathcal{A} \text{ is a } \delta- \text{ cover of } E\right\}$$

and let $D_\mu^q(E) = \limsup_{\delta\searrow 0} D_\mu^q(E,\delta)$. With these definitions Rand shows that $(1-q)\,D_\nu^q(K) = \tau(q)$ (Theorem 3 in [Ra89]).

We now consider the work of Cawley and Mauldin in their paper, *Multifractal Decompositions of Moran Fractals* [CM92]. This paper gives parallel results to those found in Rand's paper but concerns self-similar measures. Cawley and Mauldin's paper is significant in the development of multifractal analysis mainly because its simple setting allowed Cawley and Mauldin to drop the technical details of the thermodynamic formalism. In the paper Cawley and Mauldin first define an auxiliary function which they call β. This function is equivalent to the τ function which Rand defined but while it could be derived using the thermodynamic formalism, this formalism is never mentioned in Cawley and Mauldin's paper. The simplicity of their expression for β allowed Cawley and Mauldin to find many properties of β without using general results from the thermodynamic formalism and the simplicity of their analysis meant that Cawley and Mauldin's paper received a much wider reading than any previous paper in multifractal analysis. Cawley and Mauldin's analysis of β yielded two related auxiliary functions α and f (which are equivalent to the α and f functions found in Rand's paper). The main result in their paper is one that shows that f is the fine grain Hausdorff multifractal spectrum. What is interesting about their analysis is that in proving this they first show that f coincides with the Hausdorff cylinder spectrum and then use certain separation conditions to show that the cylinder spectrum coincides with the actual spectrum. The final thing which Cawley and Mauldin note in their paper is that while they prove all of their results in the map specified case they can all be extended to general Moran constructions.

We now consider Cawley and Mauldin's paper in more detail. We start with the following notations/definitions. Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be an iterated function scheme consisting of similarities defined on $\mathbf{R}^d$ with ratio list $(r_1, \ldots, r_n)$, J be a regular seed set of $\mathcal{T}$ (for convenience we assume J has diameter 1), K be the invariant set associated with $\mathcal{T}$, s be the similarity dimension of K, $p = (p_1, \ldots, p_n)$ be a probability vector and μ be the $\{\mathcal{T}, p\}$-invariant measure. In addition, for $\omega \in \{1, \cdots, n\}^{\mathbf{N}}$ and $k \in \mathbf{N}$, set $r_{\omega|k} = r_{\omega_1} \cdots r_{\omega_k}$ and $p_{\omega|k} = p_{\omega_1} \cdots p_{\omega_k}$

The first section of Cawley and Mauldin's paper is devoted to defining and deriving the properties of three auxiliary functions. The first of these is $\beta \colon \mathbf{R} \to \mathbf{R}$. For $q \in \mathbf{R}$ let $\beta(q)$ be the unique number such that,

$$\sum_{i=1}^{n} p_i^q r_i^{\beta(q)} = 1.$$

Cawley and Mauldin's second auxiliary function is α, which is given by

$$\alpha(q) = -\beta'(q).$$

Their final auxiliary function is f; for $q \in \mathbf{R}$, they set

$$f(q) = q\alpha(q) + \beta(q).$$

The following are some immediate consequences of these definitions, $\beta(1) = 0$ (since $\sum_{i=1}^{n} p_i = 1$) and $\beta(0) = s$ (since $\sum_{i=1}^{n} r_i^s = 1$). Also β is strictly decreasing and either $p_i = r_i^s$ for $i = 1, \ldots, n$ and $\beta(q) = -qs + s$ or β'' is strictly positive. Thus the function α is positive and is either equal to s for all q or strictly decreasing. Finally it follows that $f(0) = s$, $f'(q) = -q\beta''(q)$ for all $q \in \mathbf{R}$ and thus either f is equal to s for all q or f is strictly increasing between $-\infty$ to 0 and strictly decreasing between 0 to ∞.

Now let us consider the non-degenerate case *i.e.* we do not have $p_i = r_i^s$ for all i. In this situation the function $q \to \alpha(q)$ is one-one, thus for $\alpha \in [\alpha(\infty), \alpha(-\infty)]$, where $\alpha(\infty) = \lim_{q\to\infty} \alpha(q)$ and $\alpha(-\infty) = \lim_{q\to-\infty} \alpha(q)$, we can define $f(\alpha)$ to be equal to $f(q)$, where q is the real number satisfying $\alpha(q) = \alpha$. The above results tell us that f, as a function of α, is smooth and everywhere concave downwards. Later in their paper Cawley and Mauldin show that $f(\alpha)$ coincides with the Hausdorff multifractal spectrum of μ.

Cawley and Mauldin end their section on auxiliary functions by deriving the asymptotic behaviour of their functions. For $i = 1, \ldots, n$ let $a_i = \log p_i / \log r_i$ and set $\underline{a} = \min_i a_i$ and $\overline{a} = \max_i a_i$. Also let $I_{\min} = \{i \mid a_i = \underline{a}\}$ and $I_{\max} = \{i \mid a_i = \overline{a}\}$. Finally, let $e_{\min}$ be the unique number such that $\sum_{i \in I_{\min}} r_i^{e_{\min}} = 1$ and $e_{\max}$ be the unique number such that $\sum_{i \in I_{\max}} r_i^{e_{\max}} = 1$.

First, from the definition of β it is obvious that $\lim_{q\to\infty} \beta(q) = -\infty$ and $\lim_{q\to-\infty} \beta(q) = \infty$. We also have that,

$$\lim_{q\to\infty} \beta(q) + \underline{a}q = e_{\min}$$

and

$$\lim_{q\to-\infty} \beta(q) + \overline{a}q = e_{\max}$$

Thus the line $-\underline{a}q + e_{\min}$ is asymptotic to $\beta(q)$ as q tends to ∞ and the line $-\overline{a}q + e_{\max}$ is asymptotic to $\beta(q)$ as q tends to $-\infty$.

Putting all the information we have about β together gives us that the graph of β is as in Figure 10 *i.e.* β has the properties assigned to τ in part 2 of the multifractal formalism.

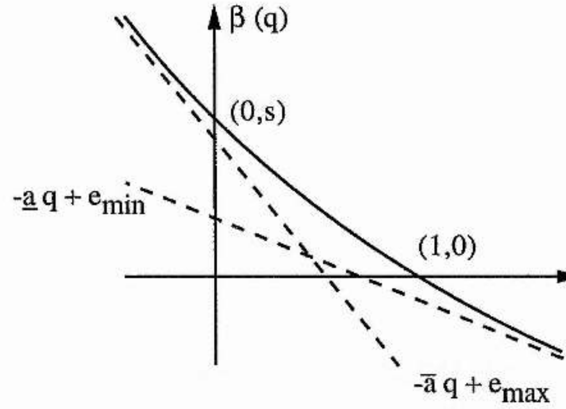


Figure 10: $\beta(q)$

Next Cawley and Mauldin go on to study the asymptotic behaviour of α. They obtain the following results,

$$\alpha(\infty) = \lim_{q\to\infty} \alpha(q) = \underline{a} = \min_i (\log p_i / \log r_i)$$

and

$$\alpha(-\infty) = \lim_{q\to-\infty} \alpha(q) = \overline{a} = \max_i (\log p_i / \log r_i).$$

The information we have about α is thus summarised by Figure 11.

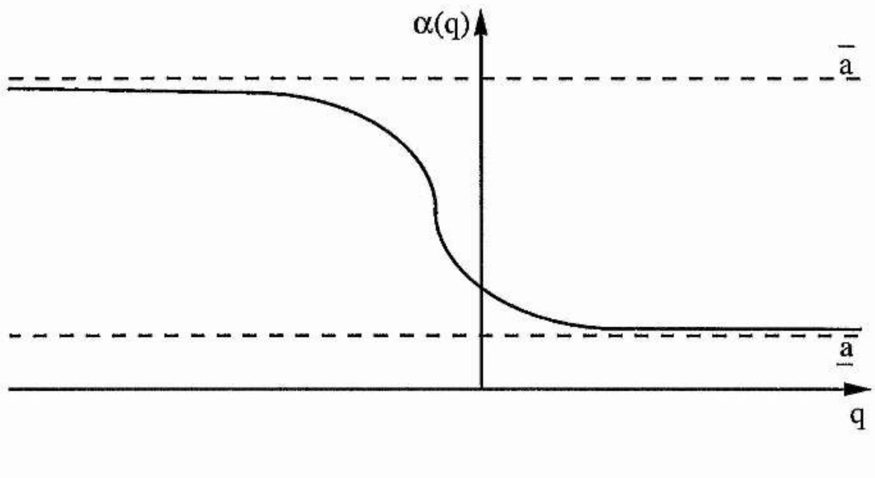


Figure 11: $\alpha(q)$

Cawley and Mauldin's investigation of the asymptotic behaviour of f yields the following results. First,

$$\lim_{q\to\infty} f(q) = e_{\min}$$

and

$$\lim_{q\to-\infty} f(q) = e_{\max}.$$

In addition they also show that f possesses the following properties:

1. $\frac{df}{d\alpha} = \alpha(q)$; in particular,

$$\lim_{\alpha \to \alpha(\infty)} \frac{df}{d\alpha} = \infty \quad \text{and} \quad \lim_{\alpha \to \alpha(-\infty)} \frac{df}{d\alpha} = -\infty.$$

2. $\frac{d^2 f}{d\alpha^2} = -\beta''(q)^{-1} < 0$; in particular $f(\alpha)$ is everywhere concave downwards.

3. $f(\alpha(1)) = \alpha(1)$; in particular, the line with slope one passing through the origin is a tangent to $f(\alpha)$ at $\alpha = \alpha(1)$.

4. The maximum value of f is given by $f_{\max} = s$.

5. $\alpha(1)$ is the Hausdorff dimension of $\hat{\mu}$ and $f(q)|_{q=1}$ is the information dimension of $\hat{\mu}$, where $\hat{\mu}$ is the invariant measure on $E^{\mathbf{N}}$ based on the probability vector $p_1, \dots, p_n$.

6. We have the following inequalities

$$e_{\min} \leq f(\alpha(1)) \leq f(\alpha(0)) = s \quad \text{and} \quad s \geq e_{\max}.$$

Thus Figure 12 illustrates the main properties of $f(\alpha)$.

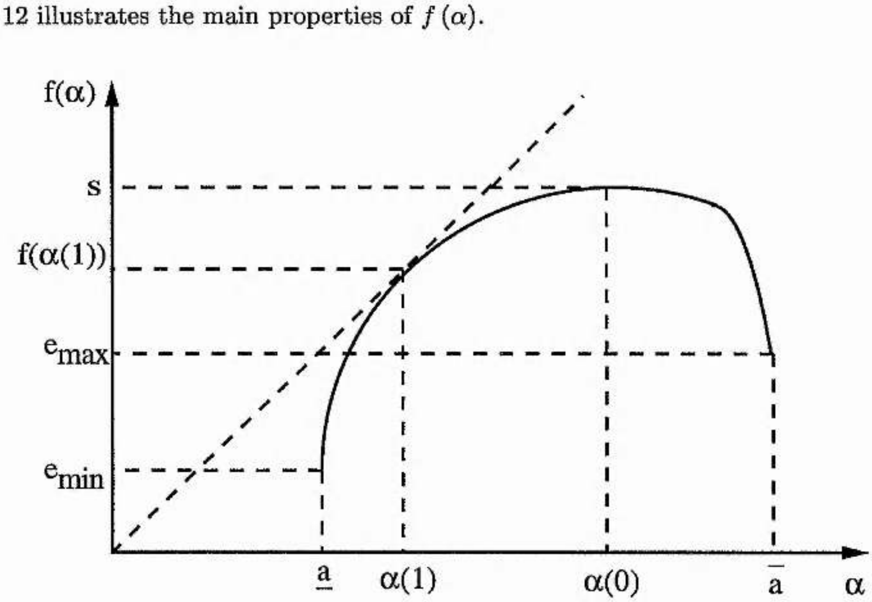


Figure 12: $f(\alpha)$

After discussing these auxiliary functions Cawley and Mauldin move on to calculate the multifractal spectrum of μ. A method frequently employed when working with self-similar or graph directed self-similar sets or measures which satisfy the strong separation condition or open set condition is to first calculate in the code space and then demonstrate that this calculation yields the same result in the geometric space. Cawley and Mauldin adopt this method in their paper.

First, given $\alpha \in \mathbf{R}$ let

$$\hat{K}_\alpha = \left\{ \omega \in E^{\mathbf{N}} \mid \lim_{k \to \infty} \log \hat{\mu}([\omega|k]) / \log r_{\omega|k} = \alpha \right\}$$

and

$$K_\alpha = \pi\left(\hat{K}_\alpha\right).$$

Thus given $q \in \mathbf{R}$ if we define $\alpha(q)$ as above then by the definition of $\hat{\mu}$,

$$K_{\alpha(q)} = \left\{ \pi(\omega) \,\middle|\, \lim_{k\to\infty} \log p_{\omega|k} / \log r_{\omega|k} = \alpha(q) \right\}.$$

Their method is to calculate the dimension of the sets K_α for each $\alpha \in \mathbf{R}$, thus generating the cylinder spectrum and then show that in the case where $\mathcal{T}$ satisfies the strong separation condition, $x \in \operatorname{supp}\mu$ has local dimension α if and only if $x \in K_\alpha$.

Recalling the definitions of a_i, $\underline{a}$ and $\overline{a}$ we have that $p_i = r_i^{a_i}$ and given $k \in \mathbf{N}$ and $\omega \in E^{\mathbf{N}}$ we have

$$\log p_{\omega|k} / \log r_{\omega|k} = \sum_{i=1}^{k} \lambda_{\omega_i} \log r_{\omega_i} / \sum_{i=1}^{k} \log r_{\omega_i}.$$

Thus for $\omega \in E^{\mathbf{N}}$

$$\begin{aligned} \underline{a} = \alpha(\infty) &\leq \liminf_{k\to\infty} \log p_{\omega|k} / \log r_{\omega|k} \\ &\leq \limsup_{k\to\infty} \log p_{\omega|k} / \log r_{\omega|k} \leq \alpha(-\infty) = \overline{a}. \end{aligned}$$

We now state one of the main theorems in Cawley and Mauldin's paper.

Theorem 4.3 *For* $\alpha \in (\alpha(\infty), \alpha(-\infty))$, $\dim_{\mathrm{H}}(K_\alpha) = f(\alpha)$ i.e. *for* $q \in \mathbf{R}$, $\dim_{\mathrm{H}}\left(K_{\alpha(q)}\right) = f(q) = q\alpha(q) + \beta(q)$.

The proof of this theorem is broken into two parts; first the upper bound *i.e.* that $\dim_{\mathrm{H}}\left(K_{\alpha(q)}\right) \leq f(q)$, which relies on a standard application of the Vitali covering theorem and then the lower bound *i.e.* that $\dim_{\mathrm{H}}\left(K_{\alpha(q)}\right) \geq f(q)$. In the lower bound calculation, for each $q \in \mathbf{R}$, Cawley and Mauldin introduce a probability measure μ_q supported on $K_{\alpha(q)}$ such that the dimension of μ_q is $f(q)$.

In turn each of these two parts is further subdivided into three cases, $q = 0$, $q < 0$ and $q > 0$. Cawley and Mauldin start by considering the upper bound theorem.

Case 1: $q = 0$

Theorem 4.4 $\dim_{\mathrm{H}}\left(K_{\alpha(0)}\right) \leq f(0) = s$.

Proof: Since $\dim_{\mathrm{H}}\left(K_{\alpha(0)}\right) \leq \dim_{\mathrm{H}}(K)$ it suffices to show that $\mathcal{H}^s(K) < \infty$. This is well known. ■

Case 2: $q > 0$ Let,

$$\hat{U}_q = \left\{ \omega \in E^{\mathbf{N}} \,\middle|\, \limsup_{k\to\infty} \log p_{\omega|k} / \log r_{\omega|k} \leq \alpha(q) \right\}$$

and

$$U_q = \pi\left(\hat{U}_q\right).$$

A standard Vitali type argument yields the following lemma:

Lemma 4.5 *For* $q, \delta > 0$ *and* $m \in \mathbf{N}$ *there exists a collection* $\mathcal{G}_m$ *of pairwise disjoint sets each with diameter less than* $1/m$ *such that*

1. $\mathcal{H}^{f(q)+\delta}\left(U_q \setminus \cup \mathcal{G}_m\right) = 0$; *and*
2. $\sum_{G\in\mathcal{G}_m} \operatorname{diam}(G)^{f(q)+\delta} \leq 1$.

From this Cawley and Mauldin deduce the following.

Theorem 4.6 *For* $q > 0$, $\dim_{\mathrm{H}}\left(K_{\alpha(q)}\right) \leq f(q)$.

Proof: Since $K_{\alpha(q)} \subseteq U_q$ it suffices to show that,

$$\mathcal{H}^{f(q)+\delta}(U_q) < \infty$$

for all $\delta > 0$. Given $q, \delta > 0$ and $m \in \mathbf{N}$ let $\mathcal{G}_m$ be the collection in Lemma 4.5. Then part (1) of Lemma 4.5 gives that $\mathcal{H}^{f(q)+\delta}(U_q \backslash \bigcap_{m=1}^{\infty} \cup \mathcal{G}_m) = 0$ and part (2) of Lemma 4.5 gives that $\mathcal{H}^{f(q)+\delta}(U_q \cap (\bigcap_{m=1}^{\infty} \cup \mathcal{G}_m)) \leq 1$. We can conclude that $\mathcal{H}^{f(q)+\delta}(U_q) \leq 1$. ■

Case 3: $q < 0$ Let,

$$\hat{L}_q = \left\{ \omega \in E^{\mathbf{N}} \,\middle|\, \liminf_{k \to \infty} \log p_{\omega|k} / \log r_{\omega|k} \geq \alpha(q) \right\}$$

and

$$L_q = \pi\left(\hat{L}_q\right).$$

Similar arguments to those in case 2 give the following theorem.

Theorem 4.7 *For $q < 0$, $\dim_{\mathrm{H}}\left(K_{\alpha(q)}\right) \leq f(q)$.*

Cawley and Mauldin then turn to the lower bounds. They start by using Kolmogorov's consistency theorem to define an infinite product measure $\hat{\mu}_q$ on the code space based on the probability vector $\left(p_1^q r_1^{\beta(q)}, \ldots, p_n^q r_n^{\beta(q)}\right)$ and then they transfer this measure to the geometric level to obtain μ_q. They then apply Birkhoff's ergodic theorem to the shift transformation, the measure $\hat{\mu}_q$ and the functions $\log p_{\omega_1}$ and $\log r_{\omega_1}$ to deduce that for $\hat{\mu}_q$-almost all ω,

$$\lim_{n \to \infty} \frac{1}{n} \log p_{\omega|n} = \sum_{i=1}^{n} (\log p_i)\, p_i^q r_i^{\beta(q)}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log r_{\omega|n} = \sum_{i=1}^{n} (\log r_i)\, p_i^q r_i^{\beta(q)},$$

for $\hat{\mu}_q$-almost all ω. Thus, for $\hat{\mu}_q$-almost all ω,

$$\lim_{n \to \infty} \frac{\log p_{\omega|n}}{\log r_{\omega|n}} = \frac{\sum_{i=1}^{n} (\log p_i)\, p_i^q r_i^{\beta(q)}}{\sum_{i=1}^{n} (\log r_i)\, p_i^q r_i^{\beta(q)}} = \alpha(q).$$

Hence, $\hat{\mu}_q\left(\hat{K}_{\alpha(q)}\right) = 1 = \mu_q\left(K_{\alpha(q)}\right)$.

Cawley and Mauldin prove the lower bounds by using the following Lemma and standard arguments to show that $\dim_{\mathrm{H}}(\mu_q) = f(q)$.

Lemma 4.8 *There exists a number $c > 0$ such that if $E \subseteq \mathbf{R}^d$ and* $\operatorname{diam}(E) < r_{\min}$ *then the cardinality of H is less than or equal to c, where*

$$H = \left\{ \tau \in E^{(*)} \mid \operatorname{diam}(J_\tau) < \operatorname{diam}(E) \leq \operatorname{diam}\left(J_{\tau||\tau|-1}\right) \quad \text{and } J_\tau \cap E \neq \emptyset \right\}.$$

The final thing which Cawley and Mauldin show in the paper is that the cylinder spectrum coincides with the actual spectrum when the iterated function scheme satisfies the strong separation condition. They achieve this by using the strong separation condition to find cylinders which approximate a ball with centre x and radius $\epsilon/2$ from inside and outside but which are within a fixed number of levels apart regardless of the size of ϵ.

4.3 Olsen's Multifractal Formalism

In this section we discuss Olsen's multifractal formalism. This formalism was motivated by Olsen's wish to provide a general mathematical setting for the ideas present in the physics literature on multifractals. His central idea was to suggest using multifractal generalisations of the centred Hausdorff and packing measure. In [Ol95] Olsen introduced these measures and several dimension functions which have similar properties to the τ/β function. He also investigated the properties of these measures and dimension functions and showed how they relate to the multifractal spectra of the measures being analysed.

We start by defining the generalised measures.

Definition 4.9 *For $q \in \mathbf{R}$ define $\varphi_q:[0,\infty) \to [0,\infty]$ by*

$$\begin{aligned} \varphi_q(x) &= \begin{cases} \infty & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } q < 0 \\ \varphi_q(x) &= 1 && \text{for } q = 0 \\ \varphi_q(x) &= \begin{cases} 0 & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } 0 < q. \end{aligned}$$

For $\mu \in \mathcal{M}^1(X)$, $E \subseteq X$, q, $t \in \mathbf{R}$ and $\delta > 0$ we make the following definitions:

$$\mathcal{H}^{q,t}_{\mu,\delta}(E) = \inf\left\{ \sum_i \varphi_q(\mu(B(x_i,r_i)))(2r_i)^t \;\middle|\; (B(x_i,r_i))_i \text{ is a centred } \delta\text{-covering of } E \right\} \quad E \neq \emptyset;$$

$$\mathcal{H}^{q,t}_{\mu,\delta}(\emptyset) = 0; \qquad \mathcal{H}^{q,t}_{\mu,0}(E) = \sup_{\delta>0} \mathcal{H}^{q,t}_{\mu,\delta}(E); \qquad \mathcal{H}^{q,t}_{\mu}(E) = \sup_{F \subseteq E} \mathcal{H}^{q,t}_{\mu,0}(F);$$

$$\mathcal{P}^{q,t}_{\mu,\delta}(E) = \sup\left\{ \sum_i \varphi_q(\mu(B(x_i,r_i)))(2r_i)^t \;\middle|\; (B(x_i,r_i))_i \text{ is a centred } \delta\text{-packing of } E \right\} \quad E \neq \emptyset;$$

$$\mathcal{P}^{q,t}_{\mu,\delta}(\emptyset) = 0; \qquad \mathcal{P}^{q,t}_{\mu,0}(E) = \inf_{\delta>0} \mathcal{P}^{q,t}_{\mu,\delta}(E); \qquad \mathcal{P}^{q,t}_{\mu}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \mathcal{P}^{q,t}_{\mu,0}(E_i).$$

Olsen first shows that these set functions are measures. Before this however we note some important features of the pre-measures. First, $\mathcal{H}^{q,t}_{\mu,0}$ is countably subadditive but not necessarily monotone. Second, $\mathcal{P}^{q,t}_{\mu,0}$ is monotone but not necessarily countably subadditive.

Proposition 4.10

1. *The set function $\mathcal{H}^{q,t}_{\mu}$ is a metric outer measure and thus a measure on the Borel algebra.*
2. *The set function $\mathcal{P}^{q,t}_{\mu}$ is a metric outer measure and thus a measure on the Borel algebra.*

Proof: See Propositions 2.2 and 2.3 in [Ol95]. ■

Olsen next shows that these measures assign a dimension in the usual way.

Proposition 4.11 *There exist unique extended real valued numbers $\dim^q_\mu(E) \in [-\infty,\infty]$, $\operatorname{Dim}^q_\mu(E) \in [-\infty,\infty]$ and $\Delta^q_\mu(E) \in [-\infty,\infty]$ such that:*

$$\begin{aligned} \mathcal{H}^{q,t}_{\mu}(E) &= \begin{cases} \infty & t < \dim^q_\mu(E) \\ 0 & \dim^q_\mu(E) < t; \end{cases} \\ \mathcal{P}^{q,t}_{\mu}(E) &= \begin{cases} \infty & t < \operatorname{Dim}^q_\mu(E) \\ 0 & \operatorname{Dim}^q_\mu(E) < t; \end{cases} \\ \mathcal{P}^{q,t}_{\mu,0}(E) &= \begin{cases} \infty & t < \Delta^q_\mu(E) \\ 0 & \Delta^q_\mu(E) < t. \end{cases} \end{aligned}$$

Proof: This follows by elementary arguments. ∎

The following properties of $\mathcal{H}_\mu^{q,t}$, $\mathcal{P}_\mu^{q,t}$and $\mathcal{P}_{\mu,0}^{q,t}$ are easily seen from the definitions: for $t \geq 0$

$$2^{-t}\mathcal{H}_\mu^{0,t} \leq \mathcal{H}^t \leq \mathcal{H}_\mu^{0,t} \ , \ \mathcal{P}_\mu^{0,t} = \mathcal{P}^t \ , \ \mathcal{P}_{\mu,0}^{0,t} = \mathcal{P}_0^t$$

where $\mathcal{H}^t$, $\mathcal{P}^t$ and $\mathcal{P}_0^t$ denote Hausdorff t-measure, packing t-measure and pre-packing t-measure respectively. Hence if we denote Hausdorff, packing and pre-packing dimension by $\dim_{\mathrm{H}}$, $\dim_{\mathrm{P}}$ and Δ respectively, then for $E \subseteq \operatorname{supp}\mu$ we have

$$\dim_{\mathrm{H}}(E) = \dim_\mu^0(E) \ , \ \dim_{\mathrm{P}}(E) = \operatorname{Dim}_\mu^0(E) \ \text{ and } \Delta(E) = \Delta_\mu^0(E).$$

An important feature of the Hausdorff and packing measures is that for $t > 0$ they satisfy $\mathcal{H}^t \leq \mathcal{P}^t$. Olsen's next step is to show a similar relationship between the generalised measures. We start by defining a subclass of measures. For $\mu \in \mathcal{M}^1(X)$ and $a > 1$ write $T_a(\mu) = \limsup_{r \searrow 0}\left(\sup_{x \in \operatorname{supp}\mu} \frac{\mu B(x,ar)}{\mu B(x,r)}\right)$ and define the family $\mathcal{M}_D^1(X)$ of *doubling probability measures* on X by

$$\mathcal{M}_D^1(X) = \left\{\mu \in \mathcal{M}^1(X) \mid T_a(\mu) < \infty \ \text{ for some } a > 1\right\}.$$

It is easily seen that the definition of $\mathcal{M}_D^1(X)$ is independent of a.

Theorem 4.12 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$ and $q, t \in \mathbf{R}$. Then*

1. $\mathcal{P}_\mu^{q,t} \leq \mathcal{P}_{\mu,0}^{q,t}$;
2. *for $0 < q$ and $\mu \in \mathcal{M}_D^1\left(\mathbf{R}^d\right)$, $\mathcal{H}_\mu^{q,t} \leq \mathcal{P}_\mu^{q,t}$;*
3. *for $q \leq 0$, $\mathcal{H}_\mu^{q,t} \leq \mathcal{P}_\mu^{q,t}$;*
4. *there exists an integer $\zeta \in \mathbf{N}$, such that $\mathcal{H}_\mu^{q,t} \leq \zeta\mathcal{P}_\mu^{q,t}$;*
5. $\dim_\mu^q \leq \operatorname{Dim}_\mu^q \leq \Delta_\mu^q$.

Proof: (1) and (5) follow from the definitions. For details concerning the rest see Proposition 2.4 in [Ol95]. We note that the ζ appearing in (4) is the ζ found in the Besicovitch covering theorem *i.e.* the covering number of $\mathbf{R}^d$. We also note that similar ideas are used in the proof of Theorem 7.4. ∎

Olsen next defines three multifractal dimension functions b_μ, B_μ and $\Lambda_\mu : \mathbf{R} \to [-\infty, \infty]$ by setting

$$b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu) \ \ B_\mu(q) = \operatorname{Dim}_\mu^q(\operatorname{supp}\mu) \ \text{ and } \Lambda_\mu(q) = \Delta_\mu^q(\operatorname{supp}\mu)$$

where $\operatorname{supp}\mu$ denotes the support of μ. The definition of these dimension functions makes it clear that they are counterparts of the τ/β function which we have discussed previously. This being the case it is important that they have the properties which physicists ascribe to them. The next theorem shows that these functions do indeed have some of these properties.

Theorem 4.13 *Let X be a metric space and $\mu \in \mathcal{M}^1(X)$, then the following hold:*

1. $\mathcal{P}_{\mu,0}^{q,t} \geq \mathcal{P}_{\mu,0}^{p,t}$ *for $q \leq p$ and* $\mathcal{P}_{\mu,0}^{q,s} \geq \mathcal{P}_{\mu,0}^{q,t}$ *for $s \leq t$.*
2. *Λ_μ is decreasing and convex.*
3. *The map $(q,t) \to \mathcal{P}_{\mu,0}^{q,t}$ is logarithmically convex* i.e. *for all $\alpha \in [0,1]$, $p, q, s, t \in \mathbf{R}$ and $E \subseteq X$,*

$$\mathcal{P}_{\mu,0}^{\alpha p+(1-\alpha)q, \alpha t+(1-\alpha)s}(E) \leq \left(\mathcal{P}_{\mu,0}^{p,t}(E)\right)^\alpha \left(\mathcal{P}_{\mu,0}^{q,s}(E)\right)^{1-\alpha}.$$

4. $\mathcal{P}_\mu^{q,t} \geq \mathcal{P}_\mu^{p,t}$ *for $q \leq p$ and* $\mathcal{P}_\mu^{q,s} \geq \mathcal{P}_\mu^{q,t}$ *for $s \leq t$.*

5. *B_μ is decreasing and convex.*

6. *$\mathcal{H}_\mu^{q,t} \geq \mathcal{H}_\mu^{p,t}$ for $q \leq p$ and $\mathcal{H}_\mu^{q,s} \geq \mathcal{H}_\mu^{q,t}$ for $s \leq t$.*

7. *b_μ is decreasing.*

8. *Let $X = \mathbf{R}^d$, $p, q \in \mathbf{R}$ and $\alpha \in [0,1]$.*

   (a) *If $\alpha p + (1-\alpha) q \leq 0$ then*

$$b_\mu (\alpha p + (1-\alpha) q) \leq \alpha B_\mu (p) + (1-\alpha) b_\mu (q).$$

   (b) *If $\alpha p + (1-\alpha) q \geq 0$ and $\mu \in M_D^1 \left(\mathbf{R}^d\right)$ then*

$$b_\mu (\alpha p + (1-\alpha) q) \leq \alpha B_\mu (p) + (1-\alpha) b_\mu (q).$$

Note: While this theorem tells us that $B_\mu (q)$ and $\Lambda_\mu (q)$ have the convexity property ascribed to them by physicists probably more interesting is the fact that $b_\mu (q)$ need not have this property (part (8) however suggests that it is close to being convex). Olsen gives an example where $b_\mu (q)$ is not convex.
Proof: See Proposition 2.10 in [Ol95]. Also, this is a special case of Proposition 7.5. ∎

Corollary 4.14 *For $\mu \in \mathcal{M}^1 (X)$ we have:*

1. *for $q < 1$, $0 \leq b_\mu (q) \leq B_\mu (q) \leq \Lambda_\mu (q)$;*

2. *$b_\mu (1) = B_\mu (1) = \Lambda_\mu (1) = 0$;*

3. *for $q > 1$, $b_\mu (q) \leq B_\mu (q) \leq \Lambda_\mu (q) \leq 0$.*

Proof: This follows immediately from the above theorem and definitions. ∎

Olsen next considers the asymptotic behaviour of these dimension functions. For $\mu \in \mathcal{M}^1 (X)$ set,

$$\underline{a}_\mu := \sup_{0<q} -\frac{b_\mu (q)}{q}, \ \overline{a}_\mu := \inf_{q<0} -\frac{b_\mu (q)}{q}, \ \underline{A}_\mu := \sup_{0<q} -\frac{B_\mu (q)}{q} \ \text{ and } \ \overline{A}_\mu := \inf_{q<0} -\frac{B_\mu (q)}{q}$$

and observe that

$$\underline{A}_\mu \leq \underline{a}_\mu \ \text{ and } \ \overline{a}_\mu \leq \overline{A}_\mu.$$

Also set

$$I_+ (\mu) = \left\{ -\frac{B_\mu (q)}{q} \Big| 0 < q \right\} \ \text{ and } \ I_- (\mu) = \left\{ -\frac{B_\mu (q)}{q} \Big| q < 0 \right\}.$$

Finally, we denote the derived set of $A \subseteq X$ by A' *i.e.*

$$A' = \{x \in A \mid \forall r > 0, \ \exists y \in (B(x,r) \setminus \{x\} \cap A)\}.$$

Proposition 4.15 *For $\mu \in \mathcal{M}^1 (X)$ we have*

1. *If $\underline{A}_\mu \in I_+' (\mu)$ then the function $q \to B_\mu (q) + \underline{A}_\mu q$ is decreasing and*

$$\underline{E}_\mu := \lim_{q \to \infty} (B_\mu (q) + \underline{A}_\mu q) \geq 0.$$

2. *If $\underline{A}_\mu \notin I_+' (\mu)$ then there exists $q_0 \in \mathbf{R}$ such that for all $q \geq q_0$,*

$$B_\mu (q) = -\underline{A}_\mu q.$$

3. If $\overline{A}_\mu \in I'_-(\mu)$ then the function $q \to B_\mu(q) + \overline{A}_\mu q$ is increasing and

$$\overline{E}_\mu := \lim_{q\to\infty} \left(B_\mu(q) + \underline{A}_\mu q\right) \geq 0.$$

4. If $\overline{A}_\mu \notin I'_-(\mu)$ then there exists $q_1 \in \mathbf{R}$ such that for all $q < q_1$,

$$B_\mu(q) = -\overline{A}_\mu q.$$

Proof: See Proposition 2.13 in [Ol95]. ∎

In part (2) of Folklore Theorem 4.1 it was claimed that τ is decreasing, convex and smooth, has affine asymptotes as $|q| \to \infty$ and $\tau(1) = 0$. The above arguments tell us that the three dimension functions do indeed have some of these properties.

Having defined the generalised Hausdorff and packing measures and the Hausdorff, packing and pre-packing dimension functions we wish to demonstrate their usefulness by showing their connection to the multifractal spectra we introduced earlier.

Given $\mu \in \mathcal{M}^1(X)$, for $\alpha \geq 0$ set

$$\begin{aligned} \overline{K}^\alpha &= \{x \in \operatorname{supp}\mu \mid \overline{\alpha}_\mu(x) \leq \alpha\}; & \overline{K}_\alpha &= \{x \in \operatorname{supp}\mu \mid \alpha \leq \overline{\alpha}_\mu(x)\};\\ \underline{K}^\alpha &= \left\{x \in \operatorname{supp}\mu \mid \underline{\alpha}_\mu(x) \leq \alpha\right\} \quad \text{and} & \underline{K}_\alpha &= \left\{x \in \operatorname{supp}\mu \mid \alpha \leq \underline{\alpha}_\mu(x)\right\}. \end{aligned}$$

Finally, let

$$K_\alpha = \underline{K}_\alpha \cap \overline{K}^\alpha = \{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\}.$$

With these definitions we have the following theorem.

Theorem 4.16 *Let X be a metric space and $\mu \in \mathcal{M}^1(X)$. Also fix $\alpha \geq 0$, $q, t \in \mathbf{R}$ and $\delta > 0$ such that $0 \leq \alpha q + t$. Then we have the following:*

1. *(a) $\mathcal{H}^{\alpha q+t+\delta}\left(\overline{K}^\alpha\right) \leq 2^{\alpha q+\delta}\mathcal{H}^{q,t}_\mu\left(\overline{K}^\alpha\right)$ for $0 \leq q$.*

   (b) $\mathcal{H}^{\alpha q+t+\delta}\left(\underline{K}_\alpha\right) \leq 2^{\alpha q+\delta}\mathcal{H}^{q,t}_\mu\left(\underline{K}_\alpha\right)$ for $q \leq 0$.

   (c) If $0 \leq \alpha q + b_\mu(q)$ then

   $$\begin{aligned} \dim_{\mathrm{H}}\left(\overline{K}^\alpha\right) &\leq \alpha q + b_\mu(q) \qquad \text{for } 0 \leq q\\ \dim_{\mathrm{H}}\left(\underline{K}_\alpha\right) &\leq \alpha q + b_\mu(q) \qquad \text{for } q \leq 0. \end{aligned}$$

   In particular, $\dim_{\mathrm{H}}\left(\overline{K}^\alpha\right) \leq \alpha$.

   (d) If $0 \leq \alpha q + B_\mu(q)$ and $X = \mathbf{R}^d$ then

   $$\begin{aligned} \dim_{\mathrm{H}}\left(\overline{K}_\alpha\right) &\leq \alpha q + B_\mu(q) \qquad \text{for } 0 \leq q\\ \dim_{\mathrm{H}}\left(\underline{K}^\alpha\right) &\leq \alpha q + B_\mu(q) \qquad \text{for } q \leq 0. \end{aligned}$$

2. *(a) $\mathcal{P}^{\alpha q+t+\delta}\left(\overline{K}^\alpha\right) \leq 2^{\alpha q+\delta}\mathcal{P}^{q,t}_\mu\left(\overline{K}^\alpha\right)$ for $0 \leq q$.*

   (b) $\mathcal{P}^{\alpha q+t+\delta}\left(\underline{K}_\alpha\right) \leq 2^{\alpha q+\delta}\mathcal{P}^{q,t}_\mu\left(\underline{K}_\alpha\right)$ for $q \leq 0$.

   (c) If $0 \leq \alpha q + B_\mu(q)$ then

   $$\begin{aligned} \dim_{\mathrm{P}}\left(\overline{K}^\alpha\right) &\leq \alpha q + B_\mu(q) \qquad \text{for } q \leq 0\\ \dim_{\mathrm{P}}\left(\underline{K}_\alpha\right) &\leq \alpha q + B_\mu(q) \qquad \text{for } 0 \leq q. \end{aligned}$$

   In particular, $\dim_{\mathrm{P}}\left(\overline{K}^\alpha\right) \leq \alpha$.

3. *(a) If $A \subseteq \overline{K}^\alpha$ is Borel then $\mathcal{H}^{q,t}_\mu(A) \leq 2^t\mathcal{H}^{\alpha q+t-\delta}(A)$ for $q \leq 0$.*

(b) *If* $A \subseteq \underline{K}_\alpha$ *is Borel then* $\mathcal{H}_\mu^{q,t}(A) \le 2^t \mathcal{H}^{\alpha q+t-\delta}(A)$ *for* $0 \le q$. *In particular, if* $A \subseteq \underline{K}_\alpha$ *is Borel and* $\mu(A) > 0$ *then* $\alpha \le \dim_{\mathrm{H}}(A)$.

4. (a) *If* $A \subseteq \overline{K}^\alpha$ *is Borel then* $\mathcal{P}_\mu^{q,t}(A) \le 2^t \mathcal{P}^{\alpha q+t-\delta}(A)$ *for* $q \le 0$.

   (b) *If* $A \subseteq \underline{K}_\alpha$ *is Borel then* $\mathcal{P}_\mu^{q,t}(A) \le 2^t \mathcal{P}^{\alpha q+t-\delta}(A)$ *for* $0 \le q$. *In particular, if* $A \subseteq \underline{K}_\alpha$ *is Borel and* $\mu(A) > 0$ *then* $\alpha \le \dim_{\mathrm{P}}(A)$.

Proof: See Propositions 2.5 through 2.8 in [Ol95]. Also, this is a special case of Theorem 7.7. ∎

These results allow us to consider the relationship between the dimension functions b_μ and B_μ and the spectra functions f_μ and F_μ. We start by giving an upper bound theorem. Given $f, g : \mathbf{R} \to \mathbf{R}$ let

$$f \sqcup g = f.1_{(-\infty,0)} + (f(0) \vee g(0)).1_{\{0\}} + g.1_{(0,\infty)}.$$

We have the following theorem.

Theorem 4.17 *Let* X *be a metric space,* $\mu \in \mathcal{M}^1(X)$ *and* $\alpha \ge 0$. *Then the following hold:*

1. $\underline{a}_\mu \le \inf \overline{\alpha}_\mu(x) \le \sup \overline{\alpha}_\mu(x) \le \overline{A}_\mu$ *and* $\underline{A}_\mu \le \inf \underline{\alpha}_\mu(x) \le \sup \underline{\alpha}_\mu(x) \le \overline{a}_\mu$;

2.
$$\lim_{\epsilon \searrow 0} \dim_{\mathrm{H}}\left(\overline{K}_{\alpha-\epsilon} \cap \overline{K}^{\alpha+\epsilon}\right) = \begin{cases} \le & (B_\mu \sqcup b_\mu)^*(\alpha) \quad \alpha \in (\underline{a}_\mu, \overline{A}_\mu) \\ = & 0 \quad \alpha \in [0,\infty) \setminus [\underline{a}_\mu \overline{A}_\mu]; \end{cases}$$

3.
$$\lim_{\epsilon \searrow 0} \dim_{\mathrm{H}}\left(\underline{K}_{\alpha-\epsilon} \cap \underline{K}^{\alpha+\epsilon}\right) = \begin{cases} \le & (b_\mu \sqcup B_\mu)^*(\alpha) \quad \alpha \in (\underline{A}_\mu, \overline{a}_\mu) \\ = & 0 \quad \alpha \in [0,\infty) \setminus [\underline{A}_\mu, \overline{a}_\mu] \end{cases}$$

4.
$$\dim_{\mathrm{H}}(K_\alpha) = \begin{cases} \le & b_\mu^*(\alpha) \quad \alpha \in (\underline{a}_\mu, \overline{a}_\mu) \\ = & 0 \quad \alpha \in [0,\infty) \setminus [\underline{a}_\mu, \overline{a}_\mu]; \end{cases}$$

5.
$$\dim_{\mathrm{P}}(K_\alpha) = \begin{cases} \le & B_\mu^*(\alpha) \quad \alpha \in (\underline{a}_\mu, \overline{a}_\mu) \\ = & 0 \quad \alpha \in [0,\infty) \setminus [\underline{a}_\mu, \overline{a}_\mu]. \end{cases}$$

Proof: This theorem follows immediately from Theorem 4.16 and the following lemma. ∎

Lemma 4.18 *If* X *is a metric space,* $\mu \in \mathcal{M}^1(X)$ *and* $\alpha \ge 0$, *then*

1. $\underline{K}^\alpha = \emptyset$ *for* $\alpha < \underline{A}_\mu$.
2. $\underline{K}_\alpha = \emptyset$ *for* $\overline{a}_\mu < \alpha$.
3. $\overline{K}_\alpha = \emptyset$ *for* $\overline{A}_\mu < \alpha$.
4. $\overline{K}^\alpha = \emptyset$ *for* $\alpha < \underline{a}_\mu$.

Proof: See Lemma 4.4 in [Ol95]. ∎

We now turn to lower bound theorems. In [Ol95] the lower bound theorem which Olsen quotes proves that one particular situation when part 3 of the multifractal formalism holds is, very roughly speaking, when the measure is a Gibbs state and its box-counting dimension exists. Although this is an important theorem, in practise the following corollaries to Theorem 4.16 are usually more useful in showing lower bounds hold *i.e.* that the multifractal formalism holds. One important thing which should be noted is that there are many measures for which the multifractal formalism does not hold (an example can be found in [Ol95]). In fact one question which several measure theorist are interested in is, can we find a necessary and sufficient condition for the multifractal formalism to hold. Another question asked by Olsen in [Ol95] is, which functions give more information about a multifractal measure, the dimension functions b_μ and B_μ or the spectra functions f_μ and F_μ? Olsen gives examples of measures where the dimension functions can be used to split measures which have the same spectra. In doing this he implicitly suggests that a return to the physicists original idea of calculating the moments of multifractal measures may be the best way to characterise them.

Theorem 4.19 *Let X be a metric space and $\mu \in \mathcal{M}^1(X)$. If $A \subseteq K_\alpha$ is a Borel set such that $\mathcal{H}^{q,t}_\mu(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\alpha q + t \geq 0$. Then,*

$$\dim_{\mathrm{H}}(A) \geq \alpha q + t.$$

In particular, if b_μ is differentiable at q and we set $\alpha(q) = -b'_\mu(q)$ then provided that $b^*(\alpha(q)) \geq 0$ and $\mathcal{H}^{q,b_\mu(q)}_\mu(K_{\alpha(q)}) > 0$ we have

$$f_\mu(\alpha(q)) = b^*_\mu(\alpha(q)).$$

Theorem 4.20 *Let X be a metric space and $\mu \in \mathcal{M}^1(X)$. If $A \subseteq K_\alpha$ is a Borel set such that $\mathcal{P}^{q,t}_\mu(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\alpha q + t \geq 0$. Then,*

$$\dim_{\mathrm{P}}(A) \geq \alpha q + t.$$

In particular, if B_μ is differentiable at q and we set $\alpha(q) = -B'_\mu(q)$ then provided that $B^*(\alpha(q)) \geq 0$ and $\mathcal{P}^{q,B_\mu(q)}_\mu(K_{\alpha(q)}) > 0$ we have

$$F_\mu(\alpha(q)) = B^*_\mu(\alpha(q)).$$

The results so far outlined give the basics of Olsen's general theory, [Ol95] also contains several other results. Most notably Olsen recalls several notions of multifractal box dimension and shows how they relate to the multifractal pre-packing dimension function and discusses generalised Rényi dimensions and their relationship to the pre-packing dimension function.

We now define the multifractal q-box counting dimensions; let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q \in \mathbf{R}$, $E \subseteq \mathbf{R}^d$ and $\delta > 0$ then set

$$S^q_{\mu,\delta}(E) = \sup\left\{\sum_i \mu(B(x_i,\delta))^q \,\middle|\, (B(x_i,\delta))_{i\in\mathbf{N}} \text{ is a centred packing of } E\right\}.$$

We define the upper and lower multifractal q-box counting dimension of E to be

$$\overline{C}^q_\mu(E) = \limsup_{\delta\searrow 0} \frac{\log S^q_{\mu,\delta}(E)}{-\log\delta}$$

and

$$\underline{C}^q_\mu(E) = \liminf_{\delta\searrow 0} \frac{\log S^q_{\mu,\delta}(E)}{-\log\delta},$$

respectively and if $\underline{C}^q_\mu(E) = \overline{C}^q_\mu(E)$ let us refer to the common value as the q-box counting dimension of E and denote it by $C^q_\mu(E)$. It is a trivial observation that for $E \subseteq \mathbf{R}^d$,

$$\underline{C}^0_\mu(E) = \underline{\dim}_{\mathrm{B}}(E)$$

and

$$\overline{C}^0_\mu(E) = \overline{\dim}_{\mathrm{B}}(E).$$

This is not the only possible way in which the multifractal box counting dimensions can be defined. Given $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q \in \mathbf{R}$, $E \subseteq \mathbf{R}^d$ and $\delta > 0$, define

$$T^q_{\mu,\delta}(E) = \inf\left\{\sum_i \mu\left(B\left(x_i,\delta\right)\right)^q \,\middle|\, \left(B\left(x_i,\delta\right)\right)_{i\in\mathbf{N}} \text{ is a centred covering of } E\right\},$$

$$\overline{L}^q_\mu(E) = \limsup_{\delta\searrow 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log\delta}$$

and

$$\underline{L}^q_\mu(E) = \liminf_{\delta\searrow 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log\delta}$$

If $\underline{L}^q_\mu(E) = \overline{L}^q_\mu(E)$ denote the common value by $L^q_\mu(E)$. The following proposition summarises the important relationships between $\underline{C}^q_\mu(E), \overline{C}^q_\mu(E)$, $\underline{L}^q_\mu(E)$, $\overline{L}^q_\mu(E)$ and $\Delta^q_\mu(E)$.

Proposition 4.21 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$ and $E \subseteq \mathbf{R}^d$. Then*

1. *For $q \le 0$, $\dim^q_\mu(E) \le \underline{L}^q_\mu(E) = \underline{C}^q_\mu(E)$ and $\overline{L}^q_\mu(E) = \overline{C}^q_\mu(E) = \Delta^q_\mu(E)$.*
2. *For $q > 0$, $\underline{L}^q_\mu(E) \le \underline{C}^q_\mu(E)$ and $\overline{L}^q_\mu(E) \le \overline{C}^q_\mu(E) \le \Delta^q_\mu(E)$.*

If in addition $\mu \in \mathcal{M}^1_D\left(\mathbf{R}^d\right)$, then for $q > 0$

$$\dim^q_\mu(E) \le \underline{L}^q_\mu(E) = \underline{C}^q_\mu(E) \text{ and } \overline{L}^q_\mu(E) = \overline{C}^q_\mu(E) = \Delta^q_\mu(E).$$

Proof: See Propositions 2.19 through 2.22 in [Ol95]. ■

Corollary 4.22 *Let $\mu \in \mathcal{M}^1_D\left(\mathbf{R}^d\right)$, then for $\alpha \in \left(\underline{a}_\mu, \overline{a}_\mu\right)$,*

$$f_\mu(\alpha) \le \inf_q\left(\alpha q + \underline{C}^q_\mu(\operatorname{supp}\mu)\right).$$

Reviewing the results in this section we see that one of the main achievements of [Ol95] is to develop a multifractal formalism which allows one to calculate the multifractal spectrum of a measure using the same methods from geometric measure theory which one uses when calculating the Hausdorff or packing dimension of a set. This achievement opens up the possibility of developing a general multifractal theory parallel to that for dimensions of sets. That is, a theory concerning product measures, projection measures, slice and intersection measures, convolutions of measures *etc.*. We finish this section by briefly discussing two papers [Ol96] and [Ol1] which discuss product measures and slice measures respectively.

Looking first at product measures we make the following definition: for $H \subseteq \mathbf{R}^{k+l}$ and $y \in \mathbf{R}^l$ let

$$H^y = \left\{x \in \mathbf{R}^k \mid (x,y) \in H\right\}.$$

Classical results (see [BM45], [Mar54], [Haa90a], [Haa90b], [HT94] and [Ho96]) give that there exists $c > 0$ such that

$$\begin{gathered}\int \mathcal{H}^s\left(H^y\right) d\mathcal{H}^t(y) \le c\mathcal{H}^{s+t}(H) \\ \mathcal{H}^{s+t}(E\times F) \le c\mathcal{H}^s(E)\,\mathcal{P}^t(F) \\ \int \mathcal{P}^s\left(H^y\right) d\mathcal{H}^t(y) \le c\mathcal{P}^{s+t}(H) \\ \text{and}\quad \mathcal{P}^{s+t}(E\times F) \le c\mathcal{P}^s(E)\,\mathcal{P}^t(F)\end{gathered}$$

for $s, t \geq 0$, $E \subseteq \mathbf{R}^k$, $F \subseteq \mathbf{R}^l$ and $H \subseteq \mathbf{R}^{k+l}$.

In [Ol96] Olsen shows that analogous results hold for the multifractal Hausdorff and packing measures. In particular, if $\mu \in \mathcal{M}_D^1\left(\mathbf{R}^k\right)$ and $\nu \in \mathcal{M}_D^1\left(\mathbf{R}^l\right)$ then for $q, s, t \in \mathbf{R}$ and $E \subseteq \mathbf{R}^k$, $F \subseteq \mathbf{R}^l$ and $H \subseteq \mathbf{R}^{k+l}$ we have that there exists $c > 0$ such that

$$\int \mathcal{H}_\mu^{q,s}\left(H^y\right) d\mathcal{H}_\nu^{q,t}(y) \leq c\mathcal{H}_{\mu\times\nu}^{q,s+t}(H)$$

$$\mathcal{H}_{\mu\times\nu}^{q,s+t}(E \times F) \leq c\mathcal{H}_\mu^{q,s}(E)\,\mathcal{P}_\nu^{q,t}(F)$$

$$\int \mathcal{P}_\mu^{q,s}\left(H^y\right) d\mathcal{H}_\nu^{q,t}(y) \leq c\mathcal{P}_{\mu\times\nu}^{q,s+t}(H)$$

$$\text{and} \quad \mathcal{P}_{\mu\times\nu}^{q,s+t}(E \times F) \leq c\mathcal{P}_\mu^{q,s}(E)\,\mathcal{P}_\nu^{q,t}(F).$$

In [Ol1] Olsen develops a rich theory concerning slice measures much of which falls outside the scope of this thesis. Here we state some simple corollaries to his results which show how the general multifractal theory which Olsen is developing closely parallels the known fractal theory. Given $m, n \in \mathbf{N}$ such that $0 \leq m \leq n$ let $G(n,m)$ denote the Grasmannian manifold of m-dimensional linear subspaces of $\mathbf{R}^n$ and let $\gamma_{n,m}$ denote the natural measure on $G(n,m)$. Also given $\Pi \in G(n,m)$ and $x \in \mathbf{R}^n$ let us set $\Pi + x = \{z + x \mid z \in \Pi\}$ and let $\Pi^\perp$ denote the orthogonal complement of Π. It is a well known result that given $\mu \in \mathcal{M}^1(\mathbf{R}^n)$ for almost all $(x, \Pi) \in \mathbf{R}^n \times G(n,m)$ a natural planar intersection measure of μ and $\mathcal{H}^m \llcorner (\Pi + x)$ exists and is unique, let us denote it by $\mu \cap (\mathcal{H}^m \llcorner (\Pi + x))$ (see [Mat75]). Finally, we define the Hausdorff co-dimension of a set $E \subseteq \mathbf{R}^n$ to be $\operatorname{codim}(E) = n - \dim_{\mathrm{H}}(E) = \dim_{\mathrm{H}}(\mathbf{R}^n) - \dim_{\mathrm{H}}(E)$. With these definitions we have the following results due to Marstrand [Mar54] and Mattila [Mat75]. Let $E \subseteq \mathbf{R}^n$ be a Borel set with Hausdorff dimension t such that $0 < \mathcal{H}^t(E) < \infty$, then

1. If $n - m < \dim_{\mathrm{H}}(E)$ and q lies in a certain range then $(\mathcal{H}^t \llcorner E) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbf{R}^n \times G(n,m)$ satisfy
$$\operatorname{codim}(E \cap (\Pi + x))(q) = \operatorname{codim}(E)(q) + \operatorname{codim}(\Pi + x)(q).$$
2. If $n - m > \dim_{\mathrm{H}}(E)$ then $(\mathcal{H}^t \llcorner E) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbf{R}^n \times G(n,m)$ satisfy
$$\operatorname{codim}(E \cap (\Pi + x)) = n.$$
3. For all $\Pi \in G(n,m)$ and $\mathcal{H}^{n-m} \llcorner \Pi^\perp$-a.a. $x \in \Pi^\perp$,
$$\mathcal{H}^{t-n-m}(E \cap (\Pi + x)) < \infty,$$
in particular for $\mathcal{H}^{n-m} \llcorner \Pi^\perp$-a.a. $x \in \Pi^\perp$,
$$\operatorname{codim}(E \cap (\Pi + x)) \geq \operatorname{codim}(E) + \operatorname{codim}(\Pi + x).$$

Now given $\mu \in \mathcal{M}^1(\mathbf{R}^n)$ let us define $\operatorname{cob}_\mu : \mathbf{R} \to [-\infty, \infty]$ by

$$\operatorname{cob}_\mu = b_{\mathcal{H}^n} - b_\mu,$$

where $b_{\mathcal{H}^n}$ denotes the multifractal dimension function of $\mathcal{H}^n$. With these definitions and certain regularity conditions on μ (see [Ol1]) Olsen is able to show the following.

1. There exists an interval I such that for $q \in I$, if $n - m < \dim_{\mathrm{H}} \mu$ then $\left(\mathcal{H}_\mu^{q,t} \llcorner \operatorname{supp} \mu\right) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbf{R}^n \times G(n,m)$ satisfy
$$\operatorname{cob}_{\mu\cap(\mathcal{H}^m \llcorner (\Pi+x))}(q) = \operatorname{cob}_\mu(q) + \operatorname{cob}_{\mathcal{H}^m \llcorner (\Pi+x)}(q).$$
2. If $n - m > \dim_{\mathrm{H}} \operatorname{supp} \mu$ then $\left(\mathcal{H}_\mu^{q,t} \llcorner \operatorname{supp} \mu\right) \times \gamma_{n,m}$-a.a. $(x, \Pi) \in \mathbf{R}^n \times G(n,m)$ satisfy
$$\operatorname{cob}_{\mu\cap(\mathcal{H}^m \llcorner (\Pi+x))}(q) = \begin{cases} -\infty & q < 0 \\ n & q = 0 \\ \infty & q > 0. \end{cases}$$

3. For all $\Pi \in G(n,m)$ and $\mathcal{H}^{n-m} \llcorner \Pi^{\perp}$-a.a. $x \in \Pi^{\perp}$,

$$\mathcal{H}^{q,b_{\mu}(q)-(n-m)(1-q)}_{\mu\cap(\mathcal{H}^m \llcorner(\Pi+x))} \left(\text{supp }\left(\mu \cap \left(\mathcal{H}^m\left(\Pi + x\right)\right)\right)\right) < \infty,$$

in particular, for $\mathcal{H}^{n-m} \llcorner \Pi^{\perp}$-a.a. $x \in \Pi^{\perp}$,

$$\text{co}b_{\mu\cap(\mathcal{H}^m \llcorner(\Pi+x))} \geq \text{co}b_{\mu} + \text{co}b_{\mathcal{H}^m \llcorner(\Pi+x)}.$$

Thus we see that in these two situations the multifractal theory closely resembles the existing fractal theory.

4.4 Remarks

In this chapter we have briefly discussed multifractal analysis from the point of view of geometric measure theory. As mentioned at the beginning of this chapter the material which we have covered is only a small fraction of that written. It is impossible for us to give even a brief outline of the rest of the work done in this area but we do feel that we should mention two important research areas. First, Lau and Ngai are amongst several authors who have been considering the multifractal analysis of self-similar measures that overlap (see for example [LN1]). Second, Patzschke, Pesin and Wiess have been analysing self-conformal multifractals (see for example [Pat97] and [PW97]). Also worth note is Lévy-Vehel and Vojak's work on higher order multifractal spectrums (see for example [LV1]).

In recent years the field of multifractal analysis has engulfed several traditional problems that can be studied by considering level sets. Measure theoretic multifractal analysis is concerned with the dimension of the following sets:

$$K_{\alpha} = \left\{x \in \text{supp}\,\mu \mid \alpha_{\mu}(x) = \alpha\right\}.$$

The wider field of study is best characterised by saying that it involves the study of the dimension of sets of the following type:

$$X_{\alpha} = \left\{x \mid f(x) = \alpha\right\},$$

where f can be anyone of a number of types of function. For examples of this type of multifractal analysis we recommend that the reader consults the following papers: [Ja94], [Ja96], [FL1], [BP91] and [PS1].

5 Multifractal Density Theorems

In this chapter we analyse the multifractal Hausdorff measure and multifractal packing measure introduced by Olsen in [Ol95] as Henstock-Thomson 'variation' measures (see [He69] and [Th76]). This analysis follows Edgar's similar analysis of the Hausdorff and packing measures as Henstock-Thomson 'variation' measures (see [Ed94] and [Ed95]). By showing that the multifractal Hausdorff and packing measures can be expressed as Henstock-Thomson 'variation' measures we are able to prove density theorems for these measures which are more refined than those found in [Ol95].

5.1 Preliminaries

Our first step is to introduce the notation that we require for 'variation' measures. A function $h : \mathbf{R}^d \times [0,\infty) \to \mathbf{R}$ is called a *variation function* on $\mathbf{R}^d$ if for each $(x,r) \in \mathbf{R}^d \times [0,\infty)$, $0 \le h(x,r) < \infty$. Given $E \subseteq \mathbf{R}^d$, a *centred covering of* E is a collection $\mathcal{B}$ of closed balls with centres in E such that $E \subseteq \cup_{\mathcal{B}} B(x,r)$, where $B(x,r)$ denotes the closed ball with centre x and radius r. A *Vitali covering of* E is a centred covering $\mathcal{V}$ of E such that for each $x \in E$ and $\epsilon > 0$, there exists $B(x,r) \in \mathcal{V}$ with $0 < r < \epsilon$. A *packing* Π *of* E is a disjoint collection of closed balls with centres in E. A *gauge on* E is a function $\Phi : E \to (0,\infty)$. Given a gauge Φ on E, a packing Π of E is said to be Φ*-fine* if and only if for all $B(x,r) \in \Pi$, $2r \le \Phi(x)$. Finally, given a variation function h and a set $E \subseteq \mathbf{R}^d$, set $(h1_E)(x,r) = h(x,r)\,1_E(x)$, where 1_E denotes the indicator function on E.

Given a Vitali covering $\mathcal{V}$ of $\mathbf{R}^d$, and a variation function h on $\mathbf{R}^d$, set

$$H_{\mathcal{V}}(h) = \sup_{\Pi} \sum_{B(x,r)\in\Pi} h(x,r),$$

where the supremum is taken over all packings $\Pi \subseteq \mathcal{V}$. The *Vitali variation* of a variation function h is

$$H_*(h) = \inf_{\mathcal{V}} H_{\mathcal{V}}(h),$$

where the infimum is taken over all Vitali coverings of $\mathbf{R}^d$. If h has the special form $h(x,r) = f(x)\,\mu(B(x,r))^q\,(2r)^t$, for some nonnegative $f : \mathbf{R}^d \to \mathbf{R}$, $q,t \in \mathbf{R}$ and $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, then set $H^{q,t}_{\mu,\mathcal{V}}(f) = H_{\mathcal{V}}(h)$ and $H^{q,t}_{\mu,*}(f) = H_*(h)$.

Given a variation function h on $\mathbf{R}^d$, and a gauge function Φ on $\mathbf{R}^d$, set

$$P_\Phi(h) = \sup_{\Pi} \sum_{B(x,r)\in\Pi} h(x,r),$$

where the supremum is taken over all Φ-fine packings of $\mathbf{R}^d$. Also, set

$$P_*(h) = \inf_{\Phi} P_\Phi(h),$$

where the infimum is taken over all gauges on $\mathbf{R}^d$. If $h(x,r) = f(x)\,\mu(B(x,r))^q\,(2r)^t$, for some nonnegative $f : \mathbf{R}^d \to \mathbf{R}$, $q,t \in \mathbf{R}$ and $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, then set $P^{q,t}_{\mu,\Phi}(f) = P_\Phi(h)$ and $P^{q,t}_{\mu,*}(f) = P_*(h)$.

In addition to this notation we require the following three theorems

Theorem 5.1 *Let μ be a non-negative Borel measure on $\mathbf{R}^d$, μ^* denote the outer measure associated with μ, $E \subseteq \mathbf{R}^d$ and $\mathcal{V}$ be a centred Vitali covering of E. Then there exists a (finite or infinite) packing $\Pi = \{B_i := B(x_i,r_i)\}_i \subseteq \mathcal{V}$ such that:*

$$\mu^*\left(E \backslash \bigcup_i B_i\right) = 0.$$

Proof: See Theorem 3.2 and remark (3) in [Gu75]. ∎

Theorem 5.2 *Let h be a variation function on $\mathbf{R}^d$ and define μ by setting $\mu(E) = H_*(h1_E)$ for all $E \subseteq \mathbf{R}^d$. Then μ is a metric outer measure on $\mathbf{R}^d$.*

Proof: See [Th76]. ∎

Given $\mu, \nu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q, t \in \mathbf{R}$ and $x \in \operatorname{supp}\mu$, we define the *upper and lower (q,t)-density of ν at x w.r.t. μ* by

$$\overline{d}_\mu^{q,t}(x,\nu) = \limsup_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} \quad \text{and} \quad \underline{d}_\mu^{q,t}(x,\nu) = \liminf_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t},$$

respectively.

Proposition 5.3 *For $\mu, \nu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$ and $q, t \in \mathbf{R}$, the function $\overline{d}_\mu^{q,t}(\cdot,\nu)$ is Borel.*

Proof: See [Rud66] for a similar argument. ∎

5.2 Multifractal Variation Measures

In this section we show that Olsen's multifractal Hausdorff and packing measures can be expressed as 'variation' measures.

Theorem 5.4 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q, t \in \mathbf{R}$ and $E \subseteq \mathbf{R}^d$, then*

$$H_{\mu,*}^{q,t}(1_E) = \mathcal{H}_\mu^{q,t}(E).$$

Proof: (a) First we verify that $\mathcal{H}_\mu^{q,t}(E) \le H_{\mu,*}^{q,t}(1_E)$. Let $F \subseteq E$ and let assume that $H_{\mu,*}^{q,t}(1_F) < \infty$. Let $\mathcal{V}$ be a Vitali cover of F such that $H_{\mu,\mathcal{V}}^{q,t}(1_F) < \infty$ and let $\epsilon > 0$ be given. Using Theorem 5.1 we can conclude that there exists a packing $(B_i := B(x_i, r_i))_i \subseteq \mathcal{V}$ such that for each i, $r_i \le \epsilon$ and $\mathcal{H}_\mu^{q,t}(F \backslash \bigcup_i B_i) = 0$. This implies that for all $\delta > 0$, $\mathcal{H}_{\mu,\delta}^{q,t}(F \backslash \bigcup_i B_i) = 0$. Let $\eta > 0$ be given, then there exists a centred ϵ-covering $(B(y_i, s_i))_i$ of $F \backslash \bigcup_i B(x_i, r_i)$ such that $\sum_i \mu(B(y_i, s_i))^q (2s_i)^t \le \eta$. Now, since $(B_i, B(y_i, s_i))_i$ is a centred ϵ-covering of F, we have that

$$\mathcal{H}_{\mu,\epsilon}^{q,t}(F) \le \sum_i \mu(B_i)^q (2r_i)^t + \sum_i \mu(B(y_i, s_i))^q (2s_i)^t \le H_{\mu,\mathcal{V}}^{q,t}(1_F) + \eta.$$

Letting η and $\epsilon \searrow 0$ gives that $\mathcal{H}_{\mu,0}^{q,t}(F) \le H_{\mu,\mathcal{V}}^{q,t}(1_F)$ and by taking infima over $\mathcal{V}$ we can conclude that $\mathcal{H}_{\mu,0}^{q,t}(F) \le H_{\mu,*}^{q,t}(1_F)$ for all $F \subseteq E$. Statement (a) is obtained by taking suprema over subsets F of E.

(b) If $\mathcal{H}_\mu^{q,t}(E) = 0$ then $H_{\mu,*}^{q,t}(1_E) = 0$. Let $\epsilon > 0$ be given. For each $n \in \mathbf{N}\backslash\{0\}$ we have that $\mathcal{H}_{\mu,\frac{1}{n}}^{q,t}(E) = 0$ thus there exists a centred covering $(B_{i,n} := B(x_{i,n}, r_{i,n}))_i$ of E such that, $x_{i,n} \in E$, $r_{i,n} < \frac{1}{n}$ and $\sum_i \mu(B_{i,n})^q (2r_{i,n})^t < \epsilon/2^{n+1}$. For each i and n set

$$\mathcal{V}_{i,n} = \{B(y, r_{i,n}) \mid y \in E \text{ and } \mid y - x_{i,n} \mid \le r_{i,n}\}.$$

Then $\mathcal{V} = \bigcup_{i,n} \mathcal{V}_{i,n}$ is a Vitali covering of E. Let $\Pi \subseteq \mathcal{V}$ be a packing. Since all elements of $\mathcal{V}_{i,n}$ contain the point $x_{i,n}$ there is at most one element of $\mathcal{V}_{i,n}$ in Π. Thus,

$$\sum_{B(x,r)\in\Pi} \mu(B(x,r))^q (2r)^t \le \sum_{i,n} \mu(B_{i,n})^q (2r_{i,n})^t \le \sum_n \frac{\epsilon}{2^{n+1}} = \epsilon.$$

Hence $H_{\mu,\mathcal{V}}^{q,t}(1_E) \le \epsilon$ which implies that $H_{\mu,*}^{q,t}(1_E) \le \epsilon$. Statement (b) follows by letting $\epsilon \searrow 0$.

(c) Finally, we show that $H_{\mu,*}^{q,t}(1_E) \le \mathcal{H}_\mu^{q,t}(E)$. We may assume that $\mathcal{H}_\mu^{q,t}(E) < \infty$. Let ν denote the restriction of $\mathcal{H}_\mu^{q,t}$ to E and fix $\alpha > 1$. Let

$$E_1 = \left\{x \in E \middle| \overline{d}_\mu^{q,t}(x,\nu) \le \alpha^{-3}\right\} \text{ and } E_2 = \left\{x \in E \middle| \overline{d}_\mu^{q,t}(x,\nu) > \alpha^{-3}\right\}.$$

First, let us consider E_1. For $n \in \mathbf{N}\backslash\{0\}$, set

$$F_n = \left\{ x \in E_1 \middle| \frac{\nu(B(x,r))}{\mu(B_{i,n})^q (2r_{i,n})^t} < \alpha^{-2} \text{ for all } r < \frac{1}{n} \right\}.$$

Since $\alpha^{-2} > \alpha^{-3}$, $F_n \nearrow E_1$. We now show that $\mathcal{H}_\mu^{q,t}(F_n) = 0$. Let $\epsilon < \frac{1}{n}$ then if $(B_i := B(x_i, r_i))_i$ is a cover of F_n such that for each i, $r_i \le \epsilon$, we have that

$$\sum_i \mu(B_i)^q (2r_i)^t \ge \alpha^2 \sum_i \nu(B_i) \ge \alpha^2 \nu\left(\bigcup_i B_i\right) \ge \alpha^2 \nu(F_n) = \alpha^2 \mathcal{H}_\mu^{q,t}(F_n).$$

Thus, $\mathcal{H}_{\mu,\epsilon}^{q,t}(F_n) \ge \alpha^2 \mathcal{H}_\mu^{q,t}(F_n)$. Letting $\epsilon \searrow 0$ we find that $\mathcal{H}_{\mu,0}^{q,t}(F_n) \ge \alpha^2 \mathcal{H}_\mu^{q,t}(F_n)$, which implies that $\mathcal{H}_\mu^{q,t}(F_n) \ge \alpha^2 \mathcal{H}_\mu^{q,t}(F_n)$. Now, since $\alpha > 1$ and $\mathcal{H}_\mu^{q,t}(F_n) < \infty$, we have that $\mathcal{H}_\mu^{q,t}(F_n) = 0$. This in turn implies that $\mathcal{H}_\mu^{q,t}(E_1) = 0$ which gives us, using (b), that $H_{\mu,*}^{q,t}(1_{E_1}) = 0$.

Next let us consider E_2. Since $\alpha^{-4} < \alpha^{-3}$, the set

$$\mathcal{V} = \left\{ B(x,r) \middle| x \in E_2 \text{ and } \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} > \alpha^{-4} \right\}$$

is a Vitali covering of E_2. Let $\Pi \subseteq \mathcal{V}$ be a packing of E_2, then

$$\sum_{B(x,r)\in\Pi} \mu(B(x,r))^q (2r)^t < \alpha^4 \sum_\Pi \mathcal{H}_\mu^{q,t}(B(x,r) \cap E) \le \alpha^4 \mathcal{H}_\mu^{q,t}(E).$$

Since this is true for all packings $\Pi \subseteq \mathcal{V}$, $H_{\mu,\mathcal{V}}^{q,t}(1_{E_2}) \le \alpha^4 \mathcal{H}_\mu^{q,t}(E)$, and thus $H_{\mu,*}^{q,t}(1_{E_2}) \le \alpha^4 \mathcal{H}_\mu^{q,t}(E)$.

Combining these two parts we find that,

$$H_{\mu,*}^{q,t}(1_E) \le H_{\mu,*}^{q,t}(1_{E_1}) + H_{\mu,*}^{q,t}(1_{E_2}) \le 0 + \alpha^4 \mathcal{H}_\mu^{q,t}(E).$$

Statement (c) is obtained by taking infima over $\alpha > 1$. ∎

We now prove an equivalent theorem for $\mathcal{P}_\mu^{q,t}$. We must start by looking at an equivalent definition of $\mathcal{P}_\mu^{q,t}$.

Let $\mu \in \mathcal{M}^1(\mathbf{R}^d)$, $q, t \in \mathbf{R}$, Φ be a gauge on $\mathbf{R}^d$ and $E \neq \emptyset \subseteq \mathbf{R}^d$. Set

$$\mathcal{P}_{\mu,\Phi}^{q,t}(E) = \sup\left\{ \sum_{B(x,r)\in\Pi} \mu(B(x,r))^q (2r)^t \middle| \Pi \text{ is a } \Phi\text{ -fine packing of } E \right\};$$
$$\mathcal{P}_{\mu,\Phi}^{q,t}(\emptyset) = 0; \qquad \mathcal{P}_{\mu,\bullet}^{q,t}(E) = \inf_\Phi \mathcal{P}_{\mu,\Phi}^{q,t}(E).$$

where the infimum is taken over all gauges on $\mathbf{R}^d$. We note that $\Phi(x) = \delta$ for all x is a gauge on $\mathbf{R}^d$ and that $\mathcal{P}_{\mu,\delta}^{q,t} = \mathcal{P}_{\mu,\Phi}^{q,t}$ for this gauge function. It is an easy exercise to verify that $\mathcal{P}_{\mu,\bullet}^{q,t}$ is a metric outer measure on $\mathbf{R}^d$. We have introduced it because of the following theorem which will be useful later on. It tells us that by taking infima over all gauges we can omit the last step in the definition of the packing measure.

Theorem 5.5 *Let $\mu \in \mathcal{M}^1(\mathbf{R}^d)$ and $q, t \in \mathbf{R}$; for $E \subseteq \mathbf{R}^d$,*

$$\mathcal{P}_{\mu,\bullet}^{q,t}(E) = \mathcal{P}_\mu^{q,t}(E).$$

Proof: Since the functions $\Phi(x) = \delta$ for all x are gauges, we have

$$\mathcal{P}_{\mu,0}^{q,t}(E) = \inf_\delta \mathcal{P}_{\mu,\delta}^{q,t}(E) \ge \inf_\Phi \mathcal{P}_{\mu,\Phi}^{q,t}(E) = \mathcal{P}_{\mu,\bullet}^{q,t}(E),$$

for all $E \subseteq \mathbf{R}^d$. Let $E \subseteq \bigcup_i E_i$, then

$$\mathcal{P}_{\mu,\bullet}^{q,t}(E) \leq \mathcal{P}_{\mu,\bullet}^{q,t}\left(\bigcup_i E_i\right) \leq \sum_i \mathcal{P}_{\mu,\bullet}^{q,t}(E_i) \leq \sum_i \mathcal{P}_{\mu,0}^{q,t}(E_i).$$

Taking infima over covers of E gives, $\mathcal{P}_{\mu,\bullet}^{q,t}(E) \leq \mathcal{P}_{\mu}^{q,t}(E)$.

On the other hand, suppose that Φ is a gauge on E. For each $n \in \mathbf{N}\setminus\{0\}$ set

$$E_n = \left\{x \in E \middle| \Phi(x) \geq \frac{1}{n}\right\}.$$

Then $E_n \nearrow E$ and for each n we have,

$$\mathcal{P}_{\mu,\Phi}^{q,t}(E) \geq \mathcal{P}_{\mu,\Phi}^{q,t}(E_n) \geq \mathcal{P}_{\mu,\frac{1}{n}}^{q,t}(E_n) \geq \mathcal{P}_{\mu,0}^{q,t}(E_n) \geq \mathcal{P}_{\mu}^{q,t}(E_n).$$

This implies that $\mathcal{P}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu,\Phi}^{q,t}(E)$; since this is true for all gauges, we have that $\mathcal{P}_{\mu}^{q,t}(E) \leq \mathcal{P}_{\mu,\bullet}^{q,t}(E)$. ■

Corollary 5.6 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q,t \in \mathbf{R}$ and $E \subseteq \mathbf{R}^d$, then*

$$P_{\mu,*}^{q,t}(1_E) = \mathcal{P}_{\mu}^{q,t}(E).$$

Proof: This follows immediately from Theorem 5.5 and the definitions. ■

5.3 Density Theorems

In this section we prove our density theorems. First we turn to proving the following proposition.

Proposition 5.7 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, $q,t \in \mathbf{R}$ and f be a non-negative real valued Borel function on $\mathbf{R}^d$. Then*

1.

$$H_{\mu,*}^{q,t}(f) = \int f(x)\, d\mathcal{H}_{\mu}^{q,t}(x).$$

2.

$$P_{\mu,*}^{q,t}(f) = \int f(x)\, d\mathcal{P}_{\mu}^{q,t}(x).$$

Proof: (1) Theorem 5.4 verifies the statement for indicator functions and usual methods, together with the observation that Borel sets are measurable because both $H_{\mu,*}^{q,t}$ and $\mathcal{H}_{\mu}^{q,t}$ are metric outer measures, allow us to extend this to simple functions. Now, if f is a non-negative Borel function then there exists a sequence f_n of simple functions that increase to f. If $c < 1$ then the sets

$$E_n = \{x \mid f_n(x) \geq cf(x)\}$$

increase to $\mathbf{R}^d$ and $H_{\mu,*}^{q,t}(f_n) \geq cH_{\mu,*}^{q,t}(f1_{E_n})$, so, $\lim_{n\to\infty} H_{\mu,*}^{q,t}(f_n) \geq cH_{\mu,*}^{q,t}(f)$. Letting $c \to 1$ yields that $H_{\mu,*}^{q,t}(f_n) \to H_{\mu,*}^{q,t}(f)$. Thus,

$$H_{\mu,*}^{q,t}(f) = \lim_{n\to\infty} H_{\mu,*}^{q,t}(f_n) = \lim_{n\to\infty} \int f_n(x)\, d\mathcal{H}_{\mu}^{q,t}(x) = \int f(x)\, d\mathcal{H}_{\mu}^{q,t}(x).$$

(2) Corollary 5.6 varifies the statement for indicator functions. The rest of the proof follows part (1). ■

Finally we prove our main results.

Theorem 5.8 *Let $\mu \in \mathcal{M}^1\left(\mathbf{R}^d\right)$, ν be a finite Borel measure on $\mathbf{R}^d$, $q, t \in \mathbf{R}$ and $E \subseteq \mathbf{R}^d$ be Borel. Then,*

1.

$$\nu(E) \geq \int_E \overline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{H}_\mu^{q,t}(x).$$

If in addition, $\mathcal{H}_\mu^{q,t}(E) < \infty$ and $\overline{d}_\mu^{q,t}(x,\nu) < \infty$ on E, then

$$\nu(E) = \int_E \overline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{H}_\mu^{q,t}(x).$$

2.

$$\nu(E) \geq \int_E \underline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{P}_\mu^{q,t}(x).$$

If in addition, $\mathcal{P}_\mu^{q,t}(E) < \infty$ and $\underline{d}_\mu^{q,t}(x,\nu) < \infty$ on E, then

$$\nu(E) = \int_E \underline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{P}_\mu^{q,t}(x).$$

Proof: Let $U \supseteq E$ be an open set and f be a finite Borel function such that, $0 \leq f(x) \leq \overline{d}_\mu^{q,t}(x,\nu)$ with strict inequality $f(x) < \overline{d}_\mu^{q,t}(x,\nu)$ whenever $\overline{d}_\mu^{q,t}(x,\nu) > 0$. Then

$$\mathcal{V} = \left\{ B(x,r) \,\middle|\, x \in E, B(x,r) \subseteq U \text{ and } \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} \geq f(x) \right\}$$

is a Vitali covering of E. Thus if $\Pi \subseteq \mathcal{V}$ is a packing, then

$$\sum_{B(x,r)\in\Pi} f(x)\,\mu(B(x,r))^q (2r)^t \leq \sum_{\Pi} \nu(B(x,r)) = \nu\left(\bigcup_{\Pi} B(x,r)\right) \leq \nu(U).$$

So $H_{\mu,\mathcal{V}}^{q,t}(f1_E) \leq \nu(U)$ and thus $H_{\mu,*}^{q,t}(f1_E) \leq \nu(U)$. Taking infima over U we obtain that $H_{\mu,*}^{q,t}(f1_E) \leq \nu(E)$. Since f is finite on E we have that $\int_E f(x)\,\mathcal{H}_\mu^{q,t}(dx) \leq \nu(E)$. Now by our choice of f we may conclude that,

$$\int_E \overline{d}_\mu^{q,t}(x,\nu)\,\mathcal{H}_\mu^{q,t}(dx) \leq \nu(E).$$

We begin verifying the second part of (1) by showing that $\nu \ll \mathcal{H}_\mu^{q,t}$ on E. Let $F \subseteq E$ be such that $\mathcal{H}_\mu^{q,t}(F) = 0$. Then for all $\epsilon > 0$, $\mathcal{H}_{\mu,\epsilon}^{q,t}(F) = 0$. For $n \in \mathbf{N}\setminus\{0\}$, set

$$F_n = \left\{ x \in F \,\middle|\, \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} < n \text{ for all } r < \frac{1}{n} \right\}.$$

Then, since $\overline{d}_\mu^{q,t}(x,\nu) < \infty$, $F_n \nearrow F$. Also, if $\epsilon < \frac{1}{n}$ then any centred ϵ covering $(B_i := B(x_i,r_i))_i$ of F_n satisfies,

$$\sum_i \mu(B_i)^q (2r_i)^t > \frac{1}{n}\sum_i \nu(B_i) \geq \nu(F_n).$$

Thus, $\mathcal{H}_{\mu,\epsilon}^{q,t}(F_n) > \frac{1}{n}\nu(F_n)$. Hence $\nu(F_n) = 0$ which implies that $\nu(F) = 0$.

Let $\epsilon > 0$ be given and $\mathcal{V}$ be a Vitali covering of E. Then,

$$\mathcal{V}' = \left\{ B(x,r) \in \mathcal{V} \,\middle|\, \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} \leq \overline{d}_\mu^{q,t}(x,\nu) + \epsilon \right\}$$

is also a Vitali covering of E. Thus Theorem 5.1 implies that there exists a packing $\Pi \subseteq \mathcal{V}'$ such that $\mathcal{H}_{\mu}^{q,t}\left(E\backslash \bigcup_{\Pi} B(x,r)\right) = 0$, which implies that $\nu\left(E\backslash \bigcup_{\Pi} B(x,r)\right) = 0$. Hence,

$$\sum_{\Pi} \left(\overline{d}_{\mu}^{q,t}(x,\nu) + \epsilon\right) \mu(B(x,r))^{q} (2r)^{t} \geq \nu(E).$$

Thus, $H_{\mu,\mathcal{V}}^{q,t}\left(\left(\overline{d}_{\mu}^{q,t}(\cdot,\nu) + \epsilon\right) 1_{E}\right) \geq \nu(E)$, which implies that $H_{\mu,*}^{q,t}\left(\left(\overline{d}_{\mu}^{q,t}(\cdot,\nu) + \epsilon\right) 1_{E}\right) \geq \nu(E)$. Now, since the integrand is finite, we have that $\int_{E} \overline{d}_{\mu}^{q,t}(x,\nu)\, \mathcal{H}_{\mu}^{q,t}(dx) + \epsilon \mathcal{H}_{\mu}^{q,t}(E) \geq \nu(E)$. The result follows by letting $\epsilon \searrow 0$.

(2) Let E be a Borel set, $U \supseteq E$ be an open neighbourhood of E and $0 < c < 1$. Then for each $x \in E$, it is possible to choose $\Phi(x)$ such that $0 < \Phi(x) < \operatorname{dist}\left(x, \mathbf{R}^{d}\backslash U\right)$ and

$$\frac{\nu(B(x,r))}{\mu(B(x,r))^{q} (2r)^{t}} \geq c\underline{d}_{\mu}^{q,t}(x,\nu),$$

for all $r < \Phi(x)$. These condition imply that Φ is a gauge on E. If Π is a Φ-fine packing of E then,

$$\sum_{\Pi} \underline{d}_{\mu}^{q,t}(x,\nu)\, \mu(B(x,r))^{q} (2r)^{t} \leq \frac{1}{c} \sum_{\Pi} \nu(B(x,r)) \leq \frac{1}{c} \nu(U).$$

Thus, $P_{\mu,*}^{q,t}\left(\underline{d}_{\mu}^{q,t}(\cdot,\nu) 1_{E}\right) \leq P_{\mu,\Phi}^{q,t}\left(\underline{d}_{\mu}^{q,t}(\cdot,\nu) 1_{E}\right) \leq \frac{1}{c}\nu(U)$. By choosing c and U appropriately we can conclude that, $P_{\mu,*}^{q,t}\left(\underline{d}_{\mu}^{q,t}(\cdot,\nu) 1_{E}\right) \leq \nu(E)$. This in turn implies that,

$$\int_{E} \underline{d}_{\mu}^{q,t}(x,\nu)\, d\mathcal{P}_{\mu}^{q,t}(x) \leq \nu(E).$$

We now turn to proving the second part of (2). Let $\epsilon > 0$ and let Φ be a gauge on E such that $\mathcal{P}_{\mu,\Phi}^{q,t}(E) < \infty$. Then

$$\mathcal{V} = \left\{ B(x,r) \,\middle|\, x \in E, 2r \leq \Phi(x) \text{ and } \frac{\nu(B(x,r))}{\mu(B(x,r))^{q} (2r)^{t}} \leq \underline{d}_{\mu}^{q,t}(x,\nu) + \epsilon \right\}$$

is a centred Vitali covering of E. Thus Theorem 5.1 implies that there exists a packing $\Pi \subseteq \mathcal{V}$ of E such that $\nu\left(E\backslash \bigcup_{\Pi} B(x,r)\right) = 0$. Thus,

$$\begin{aligned} \nu(E) &\leq \sum_{\Pi} \nu(B(x,r)) \leq \sum_{\Pi} \left(\underline{d}_{\mu}^{q,t}(x,\nu) + \epsilon\right) \mu(B(x,r))^{q} (2r)^{t} \\ &\leq P_{\mu,\Phi}^{q,t}\left(\underline{d}_{\mu}^{q,t}(\cdot,\nu) 1_{E}\right) + \epsilon \mathcal{P}_{\mu,\Phi}^{q,t}(E). \end{aligned}$$

Taking infima over Φ and ϵ yields,

$$\nu(E) \leq P_{\mu,*}^{q,t}\left(\underline{d}_{\mu}^{q,t}(\cdot,\nu) 1_{E}\right).$$

∎

6 A Multifractal Analysis of Graph Directed Self-Conformal Measures

In this chapter we discuss the multifractal geometry of graph directed self-conformal measures whose iterated function schemes satisfy the strong open set condition. In particular we give a detailed calculation of the spectrum of these measures by extending the ideas of Patzschke in [Pat97] and King and Geronimo in [KG92]. We also show that the generalised Hausdorff and packing measures introduced by Olsen in [Ol95] take positive and finite values at the critical dimension if the self-conformal measures satisfy the strong separation condition. Finally, we discuss some open questions associated with graph directed self-conformal measures.

6.1 Graph Directed Self-Conformal Iterated Function schemes

In this section we introduce a special type of GDIFS called a graph directed self-conformal iterated function scheme (GCIFS). We start with $G = (V, E)$, a finite directed connected graph, and with each vertex $u \in V$ we associate three sets, U_u, J_u and W_u. We choose these sets in the following way, for each $u \in V$ let U_u be an open and connected subset of $\mathbf{R}^d$, J_u be a regular compact subset of U_u and W_u be an open connected set such that $\overline{W}_u$ is compact and $J_u \subseteq W_u \subseteq \overline{W}_u \subseteq U_u$. Also, with each edge $e \in E$ let us associate a map T_e and a number $p_e \in (0,1)$ such that:

1. for some $\gamma \in (0,1)$ and all $e \in E$, $T_e \colon U_{t(e)} \to U_{i(e)}$ is a conformal $C^{1+\gamma}$ diffeomorphism such that $T_e\left(J_{t(e)}\right) \subseteq J_{i(e)}$;
2. for all $e \in E$ and $x \in U_{t(e)}$, $0 < |T_e'(x)| < 1$;
3. for all $u \in V$, $\sum_{v \in V} \sum_{e \in E_{u,v}} p_e = 1$.

We note that $|T_e'(x)|$ denotes the matrix norm of the derivative of the map T_e evaluated at x.

A collection $G = (V, E, (T_e)_{e \in E}, (p_e)_{e \in E})$ where the above conditions are satisfied is called a *graph directed self-conformal iterated function scheme with probabilities* (GCIFS with probabilities). Given $G = (V, E, (T_e)_{e \in E}, (p_e)_{e \in E})$, a GCIFS with probabilities, the triple $(V, E, (T_e)_{e \in E},)$ is called a *graph directed self-conformal iterated function scheme.* The vector of sets $(J_u)_{u \in V}$ is known as the vector of *seed sets* of G.

It follows from the definition of a GCIFS and the definition of the sets W_u that there exist numbers $r_{\min}$ and $r_{\max}$ such that $0 < r_{\min} \le |T_e'(x)| \le r_{\max} < 1$ for all $e \in E$ and $x \in W_{t(e)}$. Let $p_{\min} = \min_{e \in E} p_e$ and $p_{\max} = \max_{e \in E} p_e$. Finally, for $\tau \in E^{(*)}$, let us adopt our usual convention by setting $p_\tau = p_{\tau_1} \dots p_{\tau_{|\tau|}}$, $T_\tau = T_{\tau_1} \circ \dots \circ T_{\tau_{|\tau|}}$, $K_\tau = T_\tau(K_{t(\tau)})$ and $J_\tau = T_\tau(J_{t(\tau)})$.

If we set $X_u = J_u$ then we see that GCIFSs are examples of GDIFSs and thus we can deduce from Theorem 2.4 that given $G = (V, E, (T_e)_{e \in E}, (p_e)_{e \in E})$, a GCIFS with probabilities, there exists a unique vector $K = (K_u)_{u \in V}$ of non-empty compact subsets of $\mathbf{R}^d$ satisfying

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{u,v}} T_e(K_v). \tag{17}$$

We call these sets the *graph directed self-conformal sets* associated with G. Also it follows from Theorem 2.7 that there exists a unique vector $(\mu_u)_{u \in V}$ of probability measures satisfying

$$(\mu_u)_{u \in V} = \left(\sum_{e \in E_u} p_e . \mu_{t(e)} \circ T_e^{-1} \right)_{u \in V}.$$

We call these measures the *graph directed self-conformal measures* associated with G.

Given $(V, E, (T_e)_{e \in E})$, a GCIFS, set

$$\Delta = \min \left\{ \operatorname{dist}\left(T_e\left(J_{t(e)}\right), T_{e'}\left(J_{t(e')}\right)\right) \mid e, e' \in E,\ e \neq e',\ \text{and } i(e) = i(e') \right\}.$$

We say that G satisfies the *strong separation condition* (SSC) if $\Delta > 0$. If for all $e \in E$, $T_e(\operatorname{int} J_{t(e)}) \subseteq \operatorname{int} J_{i(e)}$ and for all $u \in V$ and $e, e' \in E_u$ such that $e \neq e'$, $T_e(\operatorname{int} J_{t(e)}) \cap T_{e'}(\operatorname{int} J_{t(e')}) = \emptyset$ then we say that G satisfies the *open set condition* (OSC). We note that in effect we are specifying our seed sets to be the closure of the open sets $(V_u)_{u\in V}$ in the OSC. Finally, if G satisfies the open set condition and for each $u \in V$, $\operatorname{int} J_u \cap K_u \neq \emptyset$, then we say that G satisfies the *strong open set condition* (SOSC).

For the remainder of Section 6.1 and during Section 6.2 and Section 6.3 let $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$ be a GCIFS with probabilities coded by a strongly connected graph and satisfying the strong open set condition. Also let $(U_u)_{u\in V}$, $(J_u)_{u\in V}$, $(W_u)_{u\in V}$, $(K_u)_{u\in V}$, $(\mu_u)_{u\in V}$ and $(\pi_u)_{u\in V}$ be respectively the sets, measures and maps associated with G appearing in the above definition. Finally, let us equip the code space $E^{\mathbf{N}}$ with the metric we introduced in Chapter 3. Our aim is to find an explicit representation of the measures $(\mu_u)_{u\in V}$ as Gibbs states. Given this aim we define two important functions which are related to the properties of the measures μ_u and their supports. First we define the *metric scale function* of G. This is the map $\psi \colon E^{\mathbf{N}} \to \mathbf{R}$ given by

$$\psi(\omega) = \log |T'_{\omega_1}(\pi(\sigma(\omega)))|.$$

The metric scale function ψ represents the local change in scale as one moves from $\pi\left(\sigma(\omega)\right) \in K_{t(\omega_1)}$ to $\pi(\omega) \in K_{i(\omega)}$ under the map T_{ω_1}.

Lemma 6.1 *The metric scale function ψ is γ-Hölder continuous.*

Proof: Let $\omega, \tau \in E^{\mathbf{N}}$, then since the T_e are $C^{1+\gamma}$ diffeomorphisms there exist constants $C_1, \ldots, C_5$ such that

$$\begin{aligned}
& \left| |T'_{\omega_1}\left(\pi_{t(\omega_1)}\left(\sigma\left(\omega\right)\right)\right)| - |T'_{\tau_1}\left(\pi_{t(\tau_1)}\left(\sigma\left(\tau\right)\right)\right)| \right| \\
\leq & \begin{cases} C_2 |\left(\pi_{t(\omega_1)}\left(\sigma\left(\omega\right)\right)\right) - \left(\pi_{t(\omega_1)}\left(\sigma\left(\tau\right)\right)\right)|^\gamma & \omega_1 = \tau_1 \\ C_1 & \omega_1 \neq \tau_1 \end{cases} \\
\leq & \begin{cases} C_2 C_3^\gamma d[\sigma\left(\omega\right), \sigma\left(\tau\right)]^\gamma & \omega_1 = \tau_1 \\ \left(C_1 / r_{\max}^\gamma\right) \cdot r_{\max}^\gamma & \omega_1 \neq \tau_1 \end{cases} \\
\leq & \begin{cases} C_2 c_{\max} C_3^\gamma d[\omega, \tau]^\gamma & \omega_1 = \tau_1 \\ C_4 d[\omega, \tau]^\gamma & \omega_1 \neq \tau_1 \end{cases} \\
\leq & C_5 d[\omega, \tau]^\gamma.
\end{aligned}$$

The constant C_1 comes from the fact that $\omega \to |T'_{\omega_1}\left(\pi_{t(\omega_1)}\left(\sigma\left(\omega\right)\right)\right)|$ is a continuous function defined on a compact metric space (this implies that its range is bounded). Now applying the mean value theorem we have

$$\left| \log |T'_{\omega_1}\left(\pi_{t(\omega_1)}\left(\sigma\left(\omega\right)\right)\right)| - \log |T'_{\tau_1}\left(\pi_{t(\tau_1)}\left(\sigma\left(\tau\right)\right)\right)| \right| \leq C_5 \frac{1}{r_{\min}} d[\omega, \tau]^\gamma = C_6 d[\omega, \tau]^\gamma,$$

where $C_6 = C_5 / r_{\min}$. ∎

The *measure scale function* of G is the map $\phi \colon E^{\mathbf{N}} \to \mathbf{R}$ given by

$$\phi\left(\omega\right) = \log p_{\omega_1}.$$

We note that ϕ is γ-Hölder continuous and represents the change in measure between a cylinder $[\tau]$ and the cylinder $[\tau\omega_1]$.

Finally we are able to find an explicit expression for the measures μ_u. From now on let $\hat{\mu}_\phi$ denote the Gibbs state of the mass distribution function ϕ. If we set $\gamma_u = \hat{\mu}_\phi\left(E_u^{\mathbf{N}}\right)$ and define $\hat{\mu}_{u,\phi} = \hat{\mu}_\phi \llcorner E_u^{\mathbf{N}} / \gamma_u$ then since both $\hat{\mu}_{u,\phi}$ and $\hat{\mu}_u$ are σ-invariant ergodic probability measures on $E_u^{\mathbf{N}}$ they coincide and we have that for $E \subseteq K_u$, $\mu_u\left(E\right) = \frac{1}{\gamma_u}\left(\hat{\mu}_\phi \circ \pi_u^{-1}\left(E\right)\right)$.

An important property of GCIFSs is that they satisfy the principle of bounded distortion *i.e.* the composition of finitely many of the maps T_e in a GCIFS does not distort the geometry of the seed sets of that GCIFS too much.

Recall that in Section 3.2 we defined $S_n\phi\left(\omega\right) = \sum_{i=0}^{n-1} \phi\left(\sigma^i\left(\omega\right)\right)$ and proved the principle of bounded variation:

Lemma 6.2 *Let $\varphi\colon E^{\mathbf{N}} \to \mathbf{R}$ be γ-Hölder continuous, then there exists $a_1 \in (0,\infty)$ such that for all $n \in \mathbf{N}$, $\tau \in E^{(n)}$ and $\omega, \alpha \in [\tau]$, $|S_n\varphi(\omega) - S_n\varphi(\alpha)| \le a_1$ or equivalently, $e^{-a_1} \le \frac{\exp(S_n\varphi(\omega))}{\exp(S_n\varphi(\alpha))} \le e^{a_1}$.*

An immediate consequence of the metric scale function being γ-Hölder continuous, and thus satisfying the principle of bounded variation, is the following lemma:

Lemma 6.3 *There exists a constant a_2 such that for all $\tau \in E^{(*)}$ and $\alpha \in E^{(*)}_{t(\tau)}$ we have*

$$a_2^{-1}\exp\left(S_{|\tau|}\psi(\tau)\right)\exp\left(S_{|\alpha|}\psi(\alpha)\right) \le \exp\left(S_{|\tau\alpha|}\psi(\tau\alpha)\right) \le a_2\exp\left(S_{|\tau|}\psi(\tau)\right)\exp\left(S_{|\alpha|}\psi(\alpha)\right).$$

Lemma 6.4 *Let CH denote convex hull. There exists an $M > 0$ with the property that for any $\omega \in E^{\mathbf{N}}$, $n \ge 1$ and $z_i \in \mathrm{CH}\left(T_{\sigma^i(\omega)|n-i}\left(W_{t(\omega_n)}\right)\right)$ for $1 \le i \le n$,*

$$|\log\prod_{i=1}^{n}|T'_{\omega_i}(z_i)| - S_n\psi(\omega)| \le M,$$

where ψ denotes the metric scale function of G.

Proof: This is equivalent to Lemma 3 in [Be88]. ∎

We are now able to use Lemma 6.4 to show that G has the property of bounded distortion.

Lemma 6.5 *Let a_2 be the constant appearing in Lemma 6.3, then there exists $a_3 \ge a_2$ such that for all $\omega \in E^{\mathbf{N}}$, $n \ge 1$ and $x, y \in W_{t(\omega_n)}$ we have*

$$a_3^{-1}|x-y|\exp\left(S_n\psi(\omega|n)\right) \le |T_{\omega|n}(x) - T_{\omega|n}(y)| \le a_3|x-y|\exp\left(S_n\psi(\omega|n)\right),$$

where $|.|$ denotes Euclidean distance.

Proof: An application of the mean value theorem gives,

$$|T_{\omega|n}(x) - T_{\omega|n}(y)| \le |x-y|\sup_{z\in\mathrm{CH}(x,y)}|T'_{\omega|n}(z)| \le |x-y|\prod_{i=1}^{n}\sup|T'_{\omega_i}(z_i)|$$

where the supremum is taken over all $z_i \in \mathrm{CH}\left(T_{\sigma^i(\omega)|n-i}\left(W_{t(\omega_n)}\right)\right)$. The upper bound now follows from Lemma 6.4. The lower bound is obtained using infima rather than suprema. ∎

The following bounds on the diameter of images of the seed sets and the invariant sets are an immediate consequence of G satisfying the property of bounded distortion.

Corollary 6.6 *For $\omega \in E^{\mathbf{N}}$ and $n \ge 1$ we have that if $B \subseteq W_{t(\omega_n)}$ then,*

$$a_3^{-1}\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}B \le \operatorname{diam}T_{\omega|n}(B) \le a_3\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}B;$$

in particular,

$$a_3^{-1}\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}J_{t(\omega_n)} \le \operatorname{diam}J_{\omega|n} \le a_3\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}J_{t(\omega_n)};$$

and

$$a_3^{-1}\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}K_{t(\omega_n)} \le \operatorname{diam}K_{\omega|n} \le a_3\exp\left(S_n\psi(\omega|n)\right)\operatorname{diam}K_{t(\omega_n)}.$$

6.2 The Auxiliary Function $\beta(q)$

In this section we introduce the auxiliary function $\beta(q)$ which is related to the multifractal spectra of μ_u via the Legendre transform. We also investigate some of its properties. In order to do this we first introduce some auxiliary measures that will be useful in defining $\beta(q)$.

For $q, \beta \in \mathbf{R}$ let $\hat{\mu}_{q,\beta}$ be the Gibbs state of $q\phi + \beta\psi$, where ϕ and ψ denote the measure and metric scale functions. Also define $P\colon \mathbf{R}^2 \to \mathbf{R}$ by $P(q,\beta) = P(q\phi+\beta\psi)$, where $P(\phi)$ denotes the topological pressure of ϕ.

Lemma 6.7 *The function $P(q,\beta)$ is real analytic,*

$$\begin{aligned} D_1 P(q,\beta) &= \int \phi \, d\hat{\mu}_{q,\beta} \\ \text{and } D_2 P(q,\beta) &= \int \psi \, d\hat{\mu}_{q,\beta}, \end{aligned}$$

where D_i denotes partial differentiation with respect to the i-th variable.

Proof: See [Ru78]. ■

We see that $D_2 P < 0$ so the implicit function theorem tells us that there exists a real analytic function $\beta(q)$ such that $P(q,\beta(q)) = 0$ for all $q \in \mathbf{R}$.

For $q \in \mathbf{R}$ let $\hat{\mu}_q$ be the Gibbs state of $q\phi + \beta(q)\psi$. We call these measures the *q-equilibrium measures.* Since $P(q,\beta(q)) = 0$ we can deduce that there exists a constant a_4 such that for all $\omega \in E^{\mathbf{N}}$ and $n \in \mathbf{N}$,

$$a_4^{-1} \exp\left(qS_n\phi(\omega) + \beta(q) S_n\psi(\omega)\right) \le \hat{\mu}_q([\omega|n]) \le a_4 \exp\left(qS_n\phi(\omega) + \beta(q) S_n\psi(\omega)\right).$$

Reinterpreting this using Lemma 6.2 and the definition of the measure scale function we see that there exists a constant A_q, depending only on q, such that for all $\tau \in E^{(*)}$,

$$A_q^{-1} p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} \le \hat{\mu}_q([\tau]) \le A_q p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)}. \tag{18}$$

An immediate consequence of Equation 18 is the following lemma.

Lemma 6.8

1. *If Γ is a maximal anti-chain of $E^{\mathbf{N}}$ (see next page for definition) then*

$$A_q^{-1} \le \sum_{\tau\in\Gamma} p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} \le A_q.$$

2. *If Γ_u is a maximal anti-chain of $E_u^{\mathbf{N}}$ then*

$$A_q^{-1}\gamma_u \le \sum_{\tau\in\Gamma_u} p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} \le A_q\gamma_u,$$

where $\gamma_u = \hat{\mu}_q\left(E_u^{\mathbf{N}}\right)$.

In Lemma 6.4 we found that there exists a constant a_2 such that for all $\tau \in E^{(*)}$ and $\alpha \in E_{t(\tau)}^{(*)}$ we have

$$a_2^{-1} \exp\left(S_{|\tau|}\psi(\tau)\right) \exp\left(S_{|\alpha|}\psi(\alpha)\right) \le \exp\left(S_{|\tau\alpha|}\psi(\tau\alpha)\right) \le a_2 \exp\left(S_{|\tau|}\psi(\tau)\right) \exp\left(S_{|\alpha|}\psi(\alpha)\right).$$

If we define

$$A_{-\beta} = \begin{cases} a_2^{-\beta(q)} & \beta(q) \ge 0 \\ a_2^{\beta(q)} & \beta(q) < 0 \end{cases} \quad \text{and} \quad A_\beta = \begin{cases} a_2^{\beta(q)} & \beta(q) \ge 0 \\ a_2^{-\beta(q)} & \beta(q) < 0, \end{cases}$$

we are able to deduce that for all $\tau \in E^{(*)}$, $\alpha \in E_{t(\tau)}^{(*)}$ and $q \in \mathbf{R}$ we have

$$\begin{aligned} A_{-\beta} p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} p_\alpha^q \exp\left(S_{|\alpha|}\psi(\alpha)\right)^{\beta(q)} &\le p_{\tau\alpha}^q \exp\left(S_{|\tau\alpha|}\psi(\tau\alpha)\right)^{\beta(q)} \\ &\le A_\beta p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} p_\alpha^q \exp\left(S_{|\alpha|}\psi(\alpha)\right)^{\beta(q)}. \end{aligned}$$

Thus we have the following lemma:

Lemma 6.9 *For each $q \in \mathbf{R}$ there exists a constant $\hat{A}_q$, depending only on q, such that for all $\tau \in E^{(*)}$ and $\alpha \in E^{(*)}_{t(\tau)}$ we have*

$$\hat{A}_q^{-1}\hat{\mu}_q([\tau])\,\hat{\mu}_q([\alpha]) \leq \hat{\mu}_q([\tau\alpha]) \leq \hat{A}_q\hat{\mu}_q([\tau])\,\hat{\mu}_q([\alpha]).$$

Our next aim is to find an explicit expression for the function $\beta(q)$. In order to do this we must prove a generalisation of Lemma 3.4 in [Gr86].

We recall that a subset Γ_u of $E_u^{(*)}$ is called a maximal anti-chain of $E_u^{(*)}$ if for each $\omega \in E_u^{\mathbf{N}}$ there exists a unique $\tau \in \Gamma_u$ such that $\tau \leq \omega$. For $u \in V$ and $r \in (0,1)$ set:

$$\begin{aligned} \Gamma_{u,r} &= \{\tau \in E_u^{(*)} \mid \exp\left(S_{|\tau|}\psi(\tau)\right) < r \leq \exp\left(S_{|\tau|-1}\psi(\tau||\tau|-1)\right)\}; \quad \hat{\Gamma}_{u,r} = \Gamma_{u,r/\ a_3\ \operatorname{diam} J_u}; \\ \Gamma_r &= \bigcup_{u\in V}\Gamma_{u,r} \text{ and } \hat{\Gamma}_r = \bigcup_{u \in V}\hat{\Gamma}_{u,r}. \end{aligned}$$

Note: The intuitive interpretation of Γ_r is the set of finite strings τ for which the map T_τ scales everything down by a factor of approximately r.

While considering these maximal anti-chains we state the following lemma, which is analogous to Lemma 2.6 in [CM92] and can be proved by volume estimating.

Lemma 6.10 *For each $u \in V$ there exists a constant $A_u > 0$ such that for all $r > 0$ and $x \in W_u$ both the number of elements $\tau \in \Gamma_{u,r}$ and $\tau \in \hat{\Gamma}_{u,r}$ with $J_\tau \cap B(x,r) \neq \emptyset$ is bounded by A_u.*

We now turn to proving the generalisation of Lemma 3.4 in [Gr86]. The first thing we require are the following measures. For $q \in \mathbf{R}$ and $u \in V$ let μ_u^q denote the projection of the measure $\hat{\mu}_q$ onto the set K_u under π_u *i.e.* for $E \subseteq K_u$, $\mu_u^q(E) = \hat{\mu}_q \circ \pi_u^{-1}(E)$.

Lemma 6.11 *For $q \in \mathbf{R}$ and $u \in V$ we have*

$$\int |\log \operatorname{dist}(x, \partial J_u)|\,d\mu_u^q(x) < \infty.$$

Proof: For each $u \in V$, by the SOSC, there exists an $\eta_u \in E_u^{(*)}$ such that $\delta_u = r_{\min}\operatorname{dist}(J_{\eta_u}, \partial J_v) > 0$. For $u \in V$ set $r_u = a_3^{-1}\exp\left(S_{|\eta_u|}\psi(\eta_u)\right) r_{\min}$ (where a_3 is the constant appearing in Lemma 6.5) and observe that $r_u < a_3^{-1} < 1$. Also, for $u \in V$, set $z_u = 1 - \hat{A}_q^{-1}\hat{\mu}_q([\eta_u])$ and observe that by Lemma 6.9, $z_u < 1$. Finally, set $z = \max_{u\in V} z_u$, $r = \min_{u \in V} r_u$, and $\delta = \min_{u\in V}\delta_u$.

Now let $\Gamma(n) = \Gamma_{r^n}$ and $G_n = \{\tau \in \Gamma(n) \mid \operatorname{dist}\left(J_\tau, \partial J_{i(\tau)}\right) \leq r^n\delta\}$. For $\alpha \in \Gamma(n)$ such that $t(\alpha) = u$ we have

$$\begin{aligned} \exp\left(S_{|\alpha\eta_u|}\psi(\alpha\eta_u)\right) &\geq a_2^{-1}\exp\left(S_{|\alpha|}\psi(\alpha)\right)\exp\left(S_{|\eta_u|}\psi(\eta_u)\right) \\ &\geq a_2^{-1} r^n r_{\min}\frac{r_u}{r_{\min}} a_3 \\ &\geq a_2^{-1} r^{n+1} a_3 \\ &\geq r^{n+1} \end{aligned}$$

and

$$\begin{aligned} \operatorname{dist}\left(J_{\alpha\eta_u}, \partial J_{i(\alpha)}\right) &\geq \operatorname{dist}\left(J_{\alpha\eta_u}, \partial J_\alpha\right) \\ &\geq a_3^{-1}\exp\left(S_{|\alpha|}\psi(\alpha)\right)\operatorname{dist}\left(J_{\eta_u}, \partial J_u\right) \\ &\geq a_3^{-1} r^n r_{\min}\operatorname{dist}\left(J_{\eta_u}, \partial J_u\right) \\ &\geq a_3^{-1} r^n \delta \\ &> r^{n+1}\delta. \end{aligned}$$

Hence the set of $\tau \in \Gamma(n+1)$ such that $\tau \geq \alpha\eta_u$ is non-empty and if $\tau \in \Gamma(n+1)$ and $\tau \geq \alpha\eta_u$ then $\tau \notin G_{n+1}$. Therefore,

$$\begin{aligned}
\sum_{\tau\in G_{n+1},\tau\geq\alpha} \hat{\mu}_q([\tau]) &= \hat{\mu}_q([\alpha]) - \sum_{\tau\in\Gamma(n+1)\setminus G_{n+1},\tau\geq\alpha} \hat{\mu}_q([\tau]) \\
&\leq \hat{\mu}_q([\alpha]) - \sum_{\tau\in\Gamma(n+1),\tau\geq\alpha\eta_u} \hat{\mu}_q([\tau]) \\
&= \hat{\mu}_q([\alpha]) - \hat{\mu}_q([\alpha\eta_u]) \\
&\leq \hat{\mu}_q([\alpha]) - \hat{A}_q^{-1}\hat{\mu}_q([\alpha])\hat{\mu}_q([\eta_u]) \\
&= z_u\hat{\mu}_q([\alpha]) \\
&\leq z\hat{\mu}_q([\alpha]).
\end{aligned}$$

Hence for all $\alpha \in \Gamma(n)$ we have $\sum_{\tau\in G_{n+1},\tau\geq\alpha} \hat{\mu}_q([\tau]) \leq z\hat{\mu}_q([\alpha])$. Now for each $\tau \in G_{n+1}$ there exist a unique $\alpha \in \Gamma(n)$ such that $\tau \geq \alpha$ and since $J_\tau \subseteq J_\alpha$ we have

$$\operatorname{dist}\left(J_\alpha, \partial J_{i(\alpha)}\right) \leq \operatorname{dist}\left(J_\tau, \partial J_{i(\tau)}\right) \leq r^{n+1}\delta < r^n\delta$$

i.e. $\alpha \in G_n$. From this we can derive that,

$$\sum_{\tau\in G_{n+1}} \hat{\mu}_q([\tau]) = \sum_{\alpha\in G_n}\sum_{\tau\in G_{n+1},\tau\geq\alpha} \hat{\mu}_q([\tau]) \leq z\sum_{\alpha\in G_n} \hat{\mu}_q([\alpha]).$$

Hence by induction,

$$\sum_{\tau\in G_n} \hat{\mu}_q([\tau]) \leq z^n.$$

Now let $x \in \bigcup_{u\in V} K_u$ be such that $\operatorname{dist}(x, \partial J_x) \leq r^n\delta$, where J_x denotes the seed set which contains x. We have that there exists an $\alpha \in \Gamma(n)$ such that $x \in J_\alpha$ and $\operatorname{dist}(x, \partial J_\alpha) \leq r^n\delta$ *i.e.* $\alpha \in G_n$. Hence for each $u \in V$,

$$\pi_u^{-1}\{x \in K_u \mid \operatorname{dist}(x, \partial J_u) \leq r^n\delta\} \subseteq \bigcup_{\alpha\in G_n} [\alpha]$$

and thus

$$\mu_u^q\{x \in K_u \mid \operatorname{dist}(x, \partial J_u) \leq r^n\delta\} \leq z^n.$$

Now if we set $A_n = \left\{x \in K_u \mid r^{n+1}\delta \leq \operatorname{dist}(x, \partial J_u) < r^n\delta\right\}$ then

$$\begin{aligned}
\int |\log \operatorname{dist}(x, \partial J_u)| d\mu_u^q(x) &\leq \sum_{n=1}^{\infty}\int_{A_n} |\log \operatorname{dist}(x, \partial J_u)| d\mu_u^q(x) \\
&\leq \sum_{n=1}^{\infty}\int_{A_n}\left[(n+1)\left(\log\frac{1}{r} + \log\frac{1}{\delta}\right)\right] d\mu_u^q(x) \\
&\leq \sum_{n=1}^{\infty}(n+1)A\int_{A_n} 1 d\mu_u^q(x) \\
&\leq A\sum_{n=1}^{\infty}(n+1)z^n \\
&= A\left[\frac{1}{(1-z)^2} - 1\right] \\
&< \infty
\end{aligned}$$

where $A = \log\frac{1}{r} + \log\frac{1}{\delta}$. ■

Corollary 6.12 *If $q \in \mathbf{R}$ and $u \in V$ then for all $\alpha, \tau \in E_u^{(*)}$ such that $\alpha \not\leq \tau$ and $\tau \not\leq \alpha$ we have $\mu_u^q(J_\alpha \cap J_\tau) = 0$. Also for all $\tau \in E_u^{(*)}$ we have $\mu_u^q(J_\tau) = \hat{\mu}_q([\tau])$.*

Proof: The second statement follows immediately from the first thus we only require to prove the first statement. Lemma 6.11 gives us that for all $u \in V$, $\int_{\partial J_u} \infty \, d\mu_u^q(x) = \int_{\partial J_u} |\log \operatorname{dist}(x, \partial J_u)| d\mu_u^q(x) < \infty$. Hence, for all $u \in V$, $\mu_u^q(\partial J_u) = 0$. Also the SOSC tells us that $J_\alpha \cap J_\tau \subseteq \partial J_\tau = T_\tau(\partial J_{t(\tau)})$. Hence $\mu_u^q(J_\alpha \cap J_\tau) \leq \mu_u^q(\partial J_\tau) \leq C_1 \mu_{t(\tau)}^q(\partial J_{t(\tau)}) = 0$. ■

Having proved this important lemma and corollary we can derive some of the properties of the function $\beta(q)$. We start by using Corollary 6.12 to find an explicit expression for $\beta(q)$.

Lemma 6.13 *Given $u \in V$ and $\epsilon > 0$, there exists a maximal anti-chain $A_{u,\epsilon}$ of $E_u^{\mathbf{N}}$ such that for all $\alpha \in A_{u,\epsilon}$, $r_{\min}\epsilon < \operatorname{diam}(K_\alpha) \leq \epsilon$. With $A_{u,\epsilon}$ defined in this way we have*

$$\begin{aligned}\beta(q) &= \lim_{\epsilon \to 0} \frac{-1}{\log \epsilon} \log \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q \\ &= \lim_{\epsilon \to 0} \frac{-1}{\log \epsilon} \log \sum_{\alpha \in A_{u,\epsilon}} p_\alpha^q.\end{aligned}$$

Proof: The existence of the maximal anti-chain follows from considering the families $E_u^{(1)}, E_u^{(2)}, \ldots$ and selecting from each family those cylinders having diameter less than or equal to ϵ whose parent cylinder has diameter exceeding ϵ. Since ϵ is greater than 0 only a finite number of such cylinders are required to cover K_u.

Now we have that there exists $A_q \in (0, \infty)$ such that

$$A_q^{-1} p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} \leq \mu_u^q(K_\alpha) \leq A_q p_\tau^q \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)}$$

and since for $\tau, \alpha \in A_{u,\epsilon}$ we have $\mu_u^q(K_\tau \cap K_\alpha) = 0$, we also have that there exists $C_1 \in (0, \infty)$ such that

$$C_1^{-1} \leq \sum_{\alpha \in A_{u,\epsilon}} \mu_u^q(K_\alpha) \leq C_1.$$

Thus we can deduce that there exists $C_2 \in (0, \infty)$ such that

$$C_2^{-1} \leq \sum_{\alpha \in A_{u,\epsilon}} p_\alpha^q \exp\left(S_{|\alpha|}\psi(\alpha)\right)^{\beta(q)} \leq C_2.$$

Since by Corollary 6.6 there exists $C_3 \in (0, \infty)$ such that

$$C_3^{-1}\epsilon \leq \exp\left(S_{|\alpha|}\psi(\alpha)\right) \leq C_3\epsilon$$

we have that there exists $C_4 \in (0, \infty)$ such that

$$C_4^{-1}\epsilon^{-\beta(q)} \leq \sum_{\alpha \in A_{u,\epsilon}} p_\alpha^q \leq C_4 \epsilon^{-\beta(q)}.$$

By definition $\mu_u = \mu_u^1$ thus from Equation 18 and Corollary 6.12 we have that $\mu_u(K_\alpha) = p_\alpha$. Hence,

$$C_4^{-1}\epsilon^{-\beta(q)} \leq \sum_{\alpha \in A_{u,\epsilon}} p_\alpha^q = \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q = \sum_{\alpha \in A_{u,\epsilon}} p_\alpha^q \leq C_4 \epsilon^{-\beta(q)}.$$

The desired results follow by taking logs, dividing by $-\log \epsilon$ and letting $\epsilon \to 0$. ■
This explicit expression for $\beta(q)$ allows us to deduce the following properties of $\beta(q)$.

Lemma 6.14 *We have that $\beta' \leq 0$ and $\beta'' \geq 0$. In fact, either we are in the degenerate case where β is a straight line or the zero's of β' and β'' can only occur at isolated points and we have that β is convex and strictly decreasing.*

Proof: Implicit differentiation gives us that,

$$\beta'(q) = -D_1P/D_2P.$$

Since $D_1P = \int \phi d\hat{\mu}_q$ we must show that this integral is less than or equal to zero. Now by the definition of $\hat{\mu}_\phi$ and the variation principle we have that $0 = h_{\hat{\mu}_\phi}(\sigma) + \int \phi d\hat{\mu}_\phi \geq h_{\hat{\mu}_q}(\sigma) + \int \phi d\hat{\mu}_q$, where $h_\mu(\sigma)$ denotes the entropy of μ which by definition is positive. Thus $\int \phi d\hat{\mu}_q \leq -h_{\hat{\mu}_q}(\sigma) \leq 0$ and we have that $\beta'(q) \leq 0$.

Now set $g_\epsilon(q) = \log \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q$. Then,

$$g'_\epsilon(q) = \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q \log \mu_u(K_\alpha) \Big/ \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q$$

and

$$g''_\epsilon(q) = e^{-2g_\epsilon(q)} \left[\sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q (\log \mu_u(K_\alpha))^2 \sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q - \left(\sum_{\alpha \in A_{u,\epsilon}} \mu_u(K_\alpha)^q \log \mu_u(K_\alpha) \right)^2 \right].$$

Also, if we let $A_{u,\epsilon} = \{\alpha_1, \dots, \alpha_m\}$ then we can rearrange to find that

$$g''_\epsilon(q) = e^{-2g_\epsilon(q)} \left[\sum_{i>j} \mu_u(\alpha_i)^q \mu_u(\alpha_j)^q (\log \mu_u(\alpha_i) - \log \mu_u(\alpha_j))^2 \right] \geq 0.$$

Thus, since $\beta(q) = \lim_{\epsilon \to 0} \left(\frac{-1}{\log \epsilon}\right) g_\epsilon(q)$ and the limit of a sequence of convex functions is a convex functions we have that $\beta''(q) \geq 0$.

Finally since β is real analytic β'' is too, so if $\beta'' = 0$ on some interval then $\beta'' \equiv 0$ and β is a straight line. ∎

6.3 The Multifractal Spectrum

In this section we will calculate the multifractal spectra of the measures μ_u. First let us make the following definitions: for $q \in \mathbf{R}$ let

$$\lambda(q) = \int \psi d\hat{\mu}_q;$$

$$\eta(q) = \int \phi d\hat{\mu}_q$$

$$\text{and} \quad \alpha(q) = \frac{\eta(q)}{\lambda(q)}.$$

The proof of Lemma 6.14 tells us that $\beta'(q) = -\alpha(q)$ and the fact that β is differentiable and convex gives us that for all $q \in \mathbf{R}$,

$$\beta^*(\alpha(q)) = q\alpha(q) + \beta(q).$$

We now look at some of the important properties of $\lambda(q)$ and $\eta(q)$, in particular their relationship with the metric and measure scale functions.

Lemma 6.15 *With $\lambda(q)$ and $\eta(q)$ defined as above:*

1. $\lim_{n\to\infty} \frac{1}{n} S_n\psi(\omega|n) = \lambda(q)$ *for* $\hat{\mu}_q$*-a.a.*$\omega \in E^{\mathbf{N}}$;
2. $\lim_{n\to\infty} \frac{1}{n} \log p_{\omega|n} = \eta(q)$ *for* $\hat{\mu}_q$*-a.a.*$\omega \in E^{\mathbf{N}}$.

Proof:
(1) Let $\omega \in E^{\mathbf{N}}$, it follows from Lemma 6.4 that

$$\lim_{n\to\infty} \frac{1}{n} S_n \psi\left(\omega|n\right)$$

exists if and only if

$$\lim_{n\to\infty} \frac{1}{n} \log\left|T'_{\omega|n}\left(\pi\left(\sigma^n\left(\omega\right)\right)\right)\right|$$

exists and that in this case the two limits coincide. Now using the chain rule we have that

$$\left|T'_{\omega|n}\left(\pi\left(\sigma^n\left(\omega\right)\right)\right)\right| = \left|T'_{\omega_1}\left(\pi\left(\sigma\left(\omega\right)\right)\right)\right|\left|T'_{\omega_2}\left(\pi\left(\sigma^2\left(\omega\right)\right)\right)\right| \ldots \left|T'_{\omega_n}\left(\pi\left(\sigma^n\left(\omega\right)\right)\right)\right|$$

hence by the ergodic theorem,

$$\frac{1}{n} \log\left|T'_{\omega|n}\left(\pi\left(\sigma^n\left(\omega\right)\right)\right)\right| = \frac{1}{n} \sum_{i=1}^{n} \log\left|T'_{\omega_i}\left(\pi\left(\sigma^i\left(\omega\right)\right)\right)\right| \to \lambda\left(q\right)$$

for $\hat{\mu}_q$-a.a.$\omega \in E^{\mathbf{N}}$.
(2) Let $\omega \in E^{\mathbf{N}}$, then we have,

$$p_{\omega|n} = p_{\omega_1} p_{\omega_2} \cdots p_{\omega_n}.$$

Hence by the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log p_{\omega|n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log p_{\omega_i} = \eta\left(q\right)$$

for $\hat{\mu}_q$-a.a.$\omega \in E^{\mathbf{N}}$. ■

Now given $u \in V$ and $\alpha \in \mathbf{R}$ let us set

$$K_{u,\alpha} = \left\{ x \in K_u \middle| \lim_{r\searrow 0} \frac{\log \mu_u\left(B\left(x,r\right)\right)}{\log r} = \alpha \right\}.$$

We now show that μ_u^q is a measure supported on $K_{u,\alpha(q)}$ with local dimension almost surely equal to $\beta^*\left(\alpha\left(q\right)\right)$.

Lemma 6.16 *Given $u \in V$ and $q \in \mathbf{R}$, for μ_u^q-a.a.$x \in K_u$*

1.

$$\lim_{r\searrow 0} \frac{\log \mu_u\left(B\left(x,r\right)\right)}{\log r} = \alpha\left(q\right),$$

thus

$$\mu_u^q\left(K_{u,\alpha(q)}\right) = \gamma_u;$$

2.

$$\lim_{r\searrow 0} \frac{\log \mu_u^q\left(B\left(x,r\right)\right)}{\log r} = q\alpha\left(q\right) + \beta\left(q\right),$$

thus

$$\dim_{\mathrm{H}} K_{u,\alpha(q)} \geq q\alpha\left(q\right) + \beta\left(q\right).$$

Proof:
(1) Given $r>0$ and $\omega \in E_u^{\mathbf{N}}$ choose $k_r(\omega) \in \mathbf{N}$ such that $\omega|k_r(\omega) \in \hat{\Gamma}_{u,r}$. Then $J_{\omega|k_r(\omega)} \subseteq B(\pi(\omega), r)$. Therefore with $k_r = k_r(\omega)$,

$$\begin{aligned}\frac{\log \mu_u(B(\pi(\omega), r))}{\log r} &\leq \frac{\log \mu_u\left(J_{\omega|k_r}\right)}{\log r} \\ &\leq \frac{\log p_{\omega|k_r}}{S_{k_r-1}\psi(\omega|k_r-1) + \log \operatorname{diam} J + \log a_3} \\ &= \frac{\log p_{\omega|k_r}}{k_r} / \frac{S_{k_r-1}\psi(\omega|k_r-1) + \log \operatorname{diam} J + \log a_3}{k_r}\end{aligned}$$

and hence,

$$\limsup_{r \searrow 0} \frac{\log \mu_u(B(\pi(\omega), r))}{\log r} \leq \alpha(q)$$

for $\hat{\mu}_q$-a.a.$\omega \in E_u^{\mathbf{N}}$. Now, since $\mu_u^q = \hat{\mu}_q \circ \pi_u^{-1}$,

$$\limsup_{r \searrow 0} \frac{\log \mu_u(B(x, r))}{\log r} \leq \alpha(q)$$

for μ_u^q-a.a.$x \in K_u$.

To get the opposite inequality we define the following functions: for $u \in V$ and $m \in \mathbf{N}$, let $d_{u,m} \colon E_u^{\mathbf{N}} \to \mathbf{R}$ be given by

$$d_{u,m}(\omega) = \operatorname{dist}\left(\pi_u(\omega), \partial J_{\omega|m}\right).$$

Lemma 6.5 gives us that

$$d_{u,m}(\omega) \geq a_3^{-1} \exp\left(S_m \psi(\omega|m)\right) d_{i(\omega_m),0}\left(\sigma^m(\omega)\right).$$

For $m = 0, 1, \ldots$, let us set $\Sigma_m = \left\{\omega \in E^{\mathbf{N}} \mid d_{i(\omega),m}(\omega) > 0\right\}$. It follows from Lemma 6.11 that $\hat{\mu}_q\left(E^{\mathbf{N}} \backslash \Sigma_m\right) = 0$ for $m = 0, 1, \ldots$. Also, if we set $\Sigma = \left\{\omega \in E^{\mathbf{N}} \mid d_{i(\omega),m}(\omega) > 0, \text{ for } m = 0, 1, \ldots\right\}$, then $\hat{\mu}_q\left(E^{\mathbf{N}} \backslash \Sigma\right) = 0$ since $\Sigma = \cap_m \Sigma_m$. Now for $0 < r < 1$ and $\omega \in \Sigma$ we are able to choose $m_r(\omega) \in \mathbf{N}$ such that

$$d_{u,m_r(\omega)+1}(\omega) \leq r < d_{u,m_r(\omega)}.$$

Then by the definition of $d_{u,m}$, $B(\pi_u(\omega), r) \subseteq J_{\omega|m_r(\omega)}$ thus with $m_r = m_r(\omega)$ we have

$$\begin{aligned}\frac{\log \mu_u(B(\pi(\omega), r))}{\log r} &\geq \frac{\log \mu_u\left(J_{\omega|m_r}\right)}{\log r} \\ &\geq \frac{\log p_{\omega|m_r}}{S_{m_r+1}\psi(\omega|m_r+1) + \log d_{i(\omega_{m_r+1}),0}\left(\sigma^{m_r+1}(\omega)\right) - \log a_3} \\ &= \frac{\log p_{\omega|m_r}}{m_r} / \frac{S_{m_r+1}\psi(\omega|m_r+1) + \log d_{i(\omega_{m_r+1}),0}\left(\sigma^{m_r+1}(\omega)\right) - \log a_3}{m_r}.\end{aligned}$$

Since for all $u \in V$, $\log d_{u,0}$ is integrable, the ergodic theorem gives us that,

$$\lim_{k \to \infty} \frac{\log d_{i(\omega_{k+1}),0}\left(\sigma^{k+1}(\omega)\right)}{k} = 0.$$

Hence,

$$\liminf_{r \searrow 0} \frac{\log \mu_u(B(\pi(\omega), r))}{\log r} \geq \alpha(q)$$

for $\hat{\mu}_q$-a.a.$\omega \in E_u^{\mathbf{N}}$. Thus,

$$\liminf_{r \searrow 0} \frac{\log \mu_u (B(x,r))}{\log r} \geq \alpha(q)$$

for μ_u^q-a.a.$x \in K_u$.

(2) This follows by similar arguments to those used in (1) if we note that

$$\mu_u^q \left(J_{\omega|k}\right) = \hat{\mu}_q ([\omega|k]) \geq A_q^{-1} p_{\omega|k}^q S_k \psi (\omega|k)^{\beta(q)}.$$

■

Having derived these technical results we now go on to find an upper bound for the multifractal spectrum.

Lemma 6.17 *For $u \in V$:*

1. *if $\beta^*(a) \geq 0$ then $\dim_{\mathrm{P}} K_{u,a} \leq \beta^*(a)$;*
2. *if $\beta^*(a) < 0$ then $K_{u,a} = \emptyset$;*
3. *$\dim_{\mathrm{P}} K_u \leq \beta(0)$.*

Proof: Let $\epsilon > 0$ and $\rho < \frac{1}{2}$ be given and for $m \in \mathbf{N}$ define

$$K_m = K_{u,a,m} := \left\{ x \in K_u \mid \rho^{n(a+\epsilon)} \leq \mu_u (B(x,\rho^n)) \leq \rho^{n(a-\epsilon)} \text{ for all } n \geq m \right\}.$$

Then $K_{u,a} \subseteq \bigcup_{m=1}^{\infty} K_m$. Let $(B(x_i,r_i))_i$ be a ρ^m-packing of K_m and for each $i \in \mathbf{N}$ let us choose $n_i \in \mathbf{N}$ such that $\rho^{n_i} \leq r_i < \rho^{n_i-1}$. Then $n_i \geq m$, $B(x_i,\rho^{n_i}) \subseteq B(x_i,r_i) \subseteq B\left(x_i,\rho^{n_i-1}\right)$ and the sequence $B(x_1,\rho^{n_1}), B(x_2,\rho^{n_2}), \ldots$ is disjoint.

(1) Case 1: $q \geq 0$. Let $q \geq 0$, then for $r > 0$ and $x \in \mathbf{R}^d$, we have

$$\mu_u (B(x,r)) \leq A_u \max\{p_\tau \mid \tau \in \Gamma_{u,r} \text{ and } J_\tau \cap B(x,r) \neq \emptyset\},$$

where A_u is the constant appearing in Lemma 6.10. For $n \in \mathbf{N}$ let us write $\Gamma(n) = \Gamma_{u,\rho^n}$ then by volume estimating we can find a constant C_1 such that for all $n \in \mathbf{N}$ and $\tau \in \Gamma(n)$,

$$\text{card}\,\{i = 1,2,\ldots \mid n_i = n \text{ and } B(x_i,\rho^{n_i}) \cap J_\tau \neq \emptyset\} \leq C_1.$$

Thus we have,

$$\begin{aligned}
\sum_{i=1}^{\infty} \text{diam}\,(B(x_i,r_i))^{aq+\beta(q)+\epsilon(1+q)} &\leq C_2 \sum_{i=1}^{\infty} \rho^{n_i(aq+\beta(q)+\epsilon(1+q))} \\
&= C_2 \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} \rho^{n_i q(a+\epsilon)} \\
&\leq C_2 \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} \mu_u (B(x_i,\rho^{n_i}))^q \\
&\leq C_2 A_u^q \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} \max\{p_\tau^q \mid \tau \in \Gamma(n_i),\ J_\tau \cap B(x_i,\rho^{n_i}) \neq \emptyset\} \\
&\leq C_2 A_u^q C_1 \sum_{n=m}^{\infty} \sum_{\tau \in \Gamma(n)} \rho^{n(\beta(q)+\epsilon)} p_\tau^q \\
&\leq C_2 A_u^q C_1 r_{\min}^{-|\beta(q)|} \sum_{n=m}^{\infty} \sum_{\tau \in \Gamma(n)} \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} p_\tau^q \rho^{n\epsilon} \\
&\leq C_2 A_u^q C_1 r_{\min}^{-|\beta(q)|} A_q \left(1 - \frac{1}{2^\epsilon}\right)^{-1},
\end{aligned}$$

where $C_2 = (2/\rho)^{aq+\beta(q)+\epsilon(1+q)}$. From this we can deduce that

$$\dim_{\mathrm{P}} K_m \leq aq + \beta(q) + \epsilon(1+q)$$

for all $m \in \mathbf{N}$. Thus,

$$\dim_{\mathrm{P}} K_{u,a} \leq aq + \beta(q) + \epsilon(1+q)$$

and it follows from the fact that $\epsilon > 0$ and $q \geq 0$ are arbitrary that

$$\dim_{\mathrm{P}} K_{u,a} \leq \inf\{aq + \beta(q) \mid q \geq 0\}.$$

Case 2: $q < 0$. Given $x \in K_u$ and $r > 0$ let us choose $\tau = \tau(x,r) \in \hat{\Gamma}_{u,r}$ such that $x \in J_\tau$. Then $\mu_u(B(x,r)) \geq \mu_u\left(J_{\tau(x,r)}\right) \geq p_{\tau(x,r)}$. Also if we set $\hat{\Gamma}(n) = \hat{\Gamma}_{u,\rho^n}$ then

$$\begin{aligned}
\sum_{i=1}^{\infty} \operatorname{diam}\left(B(x_i,r_i)\right)^{aq+\beta(q)+\epsilon(1-q)} &\leq C_3 \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} \rho^{n_i q(a-\epsilon)} \\
&\leq C_3 \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} \mu_u\left(B(x_i,\rho^{n_i})\right)^q \\
&\leq C_3 \sum_{i=1}^{\infty} \rho^{n_i(\beta(q)+\epsilon)} p^q_{\tau(x_i,r_i)} \\
&\leq C_3 \sum_{n=m}^{\infty} \sum_{\tau \in \hat{\Gamma}(n)} \rho^{n(\beta(q)+\epsilon)} p_\tau^q \\
&\leq C_3 r_{\min}^{-|\beta(q)|} \sum_{n=m}^{\infty} \sum_{\tau \in \hat{\Gamma}(n)} \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} p_\tau^q \rho^{n\epsilon} \\
&\leq C_3 r_{\min}^{-|\beta(q)|} A_q \left(1 - \frac{1}{2^\epsilon}\right)^{-1},
\end{aligned}$$

where $C_3 = (2/\rho)^{aq+\beta(q)+\epsilon(1-q)}$. Now we can use similar arguments to those used in Case 1 to show that

$$\dim_{\mathrm{P}} K_{u,a} \leq \inf\{aq + \beta(q) \mid q < 0\}.$$

Case 1 and Case 2 combine to give that

$$\dim_{\mathrm{P}} K_{u,a} \leq \inf_{q \in \mathbf{R}} \{aq + \beta(q)\} = \beta^*(a).$$

(2) If $\beta^*(a) < 0$ then by definition there exists $q \in \mathbf{R}$ such that $aq + \beta(q) < 0$. This proof can be divided into two cases, $q \geq 0$ and $q < 0$. We cover the case where $q \geq 0$, similar arguments can be used for the case $q < 0$.

Let us suppose that the q such that $aq + \beta(q) < 0$ is greater than or equal to zero and let us choose $\epsilon > 0$ such that $c = -(aq + \beta(q) + \epsilon q) > 0$. Then for $x \in K_{u,a}$ there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, $\mu_u(B(x,\rho^n)) \geq \rho^{n(a+\epsilon)}$. Hence,

$$\begin{aligned}
\rho^{-nc} &= \rho^{n(aq+\beta(q)+\epsilon q)} \\
&= \rho^{n\beta(q)} \rho^{nq(a+\epsilon)} \\
&\leq \rho^{n\beta(q)} \mu_u\left(B(x,\rho^n)\right)^q \\
&\leq A_u^q \rho^{n\beta(q)} \max\left\{p_\tau^q \mid \tau \in \Gamma(n) \text{ and } J_\tau \cap B(x,\rho^n) \neq \emptyset\right\} \\
&\leq A_u^q \sum_{\tau \in \Gamma(n)} \rho^{n\beta(q)} p_\tau^q \\
&\leq A_u^q r_{\min}^{-|\beta(q)|} \sum_{\tau \in \Gamma(n)} \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(q)} p_\tau^q \\
&\leq A_u^q r_{\min}^{-|\beta(q)|} A_q.
\end{aligned}$$

Now this last expression does not depend on n, therefore letting $n \to \infty$ we have the contradiction that $\infty \le A_u^q r_{\min}^{-|\beta(q)|} A_q$. Hence we have the result.

(3) For $m \in \mathbf{N}$ let $(B(x_i, r_i))_i$ be a ρ^m-packing of K_u. Then for $\epsilon > 0$ we have,

$$\begin{aligned}
\sum_{i=1}^{\infty} \operatorname{diam} B(x_i, r_i)^{\beta(0)+\epsilon} &\le (2/\rho)^{\beta(0)+\epsilon} \sum_{i=1}^{\infty} \rho^{n_i(\beta(0)+\epsilon)} \\
&\le (2/\rho)^{\beta(0)+\epsilon} \sum_{i=1}^{\infty} \rho^{n_i(\beta(0)+\epsilon)} \sum_{\tau \in \Gamma(n)} 1_{J_\tau \cap B(x_i, \rho^{n_i}) \ne \emptyset} \\
&\le (2/\rho)^{\beta(0)+\epsilon} C_1 \sum_{n=m}^{\infty} \sum_{\tau \in \Gamma(n)} \rho^{n(\beta(0)+\epsilon)} \\
&\le (2/\rho)^{\beta(0)+\epsilon} C_1 r_{\min}^{-|\beta(0)|} \sum_{n=m}^{\infty} \sum_{\tau \in \Gamma(n)} \exp\left(S_{|\tau|}\psi(\tau)\right)^{\beta(0)} \rho^{n\epsilon} \\
&\le (2/\rho)^{\beta(0)+\epsilon} C_1 r_{\min}^{-|\beta(0)|} C_0 \left(1 - \frac{1}{2^\epsilon}\right)^{-1}.
\end{aligned}$$

Since this expression is bounded independent of m, $\dim_{\mathrm{P}} K_u \le \beta(0) + \epsilon$ for all $\epsilon > 0$. The result follows by letting $\epsilon \searrow 0$. ∎

Finally if we set $\underline{a} = \inf_{q \in \mathbf{R}} \{\alpha(q)\}$ and $\overline{a} = \sup_{q \in \mathbf{R}} \{\alpha(q)\}$ we obtain the following theorem.

Theorem 6.18

1. *If $\underline{a} = \overline{a} = \delta$ then for all $u \in V$,*

$$\dim_{\mathrm{H}} K_u = \dim_{\mathrm{P}} K_u = \dim_{\mathrm{H}} K_{u,\delta} = \dim_{\mathrm{P}} K_{u,\delta} = \delta$$

   and for all $\alpha \ne \delta$, $K_{u,\alpha} = \emptyset$.

2. *If $\underline{a} < \overline{a}$ then for all $u \in V$,*
   (a) $\dim_{\mathrm{H}} K_{u,\alpha} = \dim_{\mathrm{P}} K_{u,\alpha} = \beta^*(\alpha)$ *for* $\alpha \in (\underline{a}, \overline{a})$.
   (b) $K_{u,\alpha} = \emptyset$ *for* $\alpha \notin (\underline{a}, \overline{a})$.
   (c) $\dim_{\mathrm{H}} K_u = \dim_{\mathrm{P}} K_u = \dim_{\mathrm{H}} K_{u,\alpha(0)} = \dim_{\mathrm{P}} K_{u,\alpha(0)} = \beta(0) = \sup_a \beta^*(a)$.

Proof:

(1) In this situation δ is the only point where β^* is positive. This gives us that $\alpha(q) = -\beta'(q) = \delta$ for all $q \in \mathbf{R}$. Now since $\beta(1) = 0$ we have that $\beta(q) = \delta(1-q)$. The assertions now follow from the previous results.

(2) Since β is differentiable and convex, $\alpha(q) = -\beta'(q)$ is continuous and non-increasing. Therefore for all $\alpha \in (\underline{a}, \overline{a})$ there exists a unique $q \in \mathbf{R}$ such that $\alpha = \alpha(q)$. The above results yield that,

$$\beta^*(\alpha) = q\alpha(q) + \beta(q) \le \dim_{\mathrm{H}} K_{u,\alpha(q)} \le \dim_{\mathrm{P}} K_{u,\alpha(q)} \le \beta^*(\alpha),$$

for all $\alpha \in (\underline{a}, \overline{a})$. The assertions now follow immediately from the above results. ∎

6.4 An Application of Olsen's Formalism

In this section we show that the multifractal measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ take positive and finite values at the critical dimension if the GCIFS we are working with satisfies the strong separation condition.

Throughout this section let $G = \left(V, E, (T_e)_{e \in E}, (p_e)_{e \in E}\right)$ be a GCIFS with probabilities satisfying the strong separation condition and based on a strongly connected graph. Also, let us adopt all of the conventions we introduced in Section 6.1. In order to prove the results in this section we require the following corollary to our results on bounded distortion:

Lemma 6.19 *There exists $a_5 \in (0,1)$ such that for all $n \in \mathbf{N}$, $\tau \in E^{(n)}$ and $e, e' \in E_{t(\tau)}$ such that $e \neq e'$,*

$$\operatorname{dist}(K_{\tau e}, K_{\tau e'}) \geq a_5 \operatorname{diam} K_\tau.$$

Proof: Let us choose x and y satisfying

$$\operatorname{dist}(T_{\tau e}(x), T_{\tau e'}(y)) = \operatorname{dist}(K_{\tau e}, K_{\tau e'}).$$

Then using Lemma 6.5 we find that

$$\begin{aligned}
\frac{\operatorname{dist}(K_{\tau e}, K_{\tau e'})}{\operatorname{diam} K_\tau} &= \frac{|T_{\tau e}(x) - T_{\tau e'}(y)|}{\operatorname{diam} K_\tau} \\
&\geq \frac{a_3^{-1}|T_e(x) - T_{e'}(y)| \exp(S_n\phi(\tau))}{a_3 \exp(S_n\phi(\tau))} \\
&= \frac{1}{a_3^2}|T_e(x) - T_{e'}(y)| \\
&\geq \frac{\Delta}{a_3^2}.
\end{aligned}$$

■

We start our calculations with the following Lemma:

Lemma 6.20 *Let a_5 be the constant in Lemma 6.19. Then there exist positive and finite constants a_6, a_7, and a_8, such that if $x \in K_u$, $r > 0$, $\omega \in E_u^{\mathbf{N}}$ and $k, l \in \mathbf{N}$ and satisfy*

$$\pi_u(\omega) = x,$$

$$\operatorname{diam}\left(K_{\omega|k+1}\right) \leq r < \operatorname{diam}\left(K_{\omega|k}\right)$$

and

$$a_5 \operatorname{diam}\left(K_{\omega|l+1}\right) \leq r < a_5 \operatorname{diam}\left(K_{\omega|l}\right),$$

then

1. $K_{\omega|k+1} \subseteq B(x,r)$;
2. $K_u \cap B(x,r) \subseteq K_{\omega|l+1}$;
3. $k - l \leq a_6$;
4. $a_7^{-1}\exp(S_k\phi(\omega|k)) \leq r < a_7 \exp(S_k\phi(\omega|k))$;
5. $a_8^{-1}\exp(S_l\phi(\omega|l)) \leq r < a_8 \exp(S_l\phi(\omega|l))$.

Proof: That $K_{\omega|k+1} \subseteq B(x,r)$ follows immediately from the definition of k.

Let us suppose that $K_u \cap B(x,r) \nsubseteq K_{\omega|l+1}$. Then there exists $y \in K_u \cap B(x,r)$ such that $y \notin K_{\omega|l+1}$. Let us choose $\sigma \in E_u^{\mathbf{N}}$ such that $\pi_u(\sigma) = y$, then since ω and σ are in $E_u^{\mathbf{N}}$ and $y \notin K_{\omega|l+1}$ there exists $j < l+1$ such that $\omega|j = \sigma|j$ and $\omega_{j+1} \neq \sigma_{j+1}$. Thus,

$$|y - x| \geq \operatorname{dist}\left(K_{\omega|j+1}, K_{\sigma|j+1}\right) \geq a_5 \operatorname{diam}\left(K_{\omega|j}\right) \geq a_5 \operatorname{diam}\left(K_{\omega|l}\right) > r.$$

This is a contradiction since $y \in B(x,r)$.

By definition $a_5 \leq 1$ thus $k \geq l$ and we have

$$1 = \frac{r}{r} \leq \frac{\operatorname{diam}\left(K_{\omega|k}\right)}{a_5 \operatorname{diam}\left(K_{\omega|l+1}\right)} \leq \frac{(r_{\max})^{k-l-1}}{a_5}.$$

Taking logarithms and rearranging we find that

$$k - l \leq 1 + \frac{\log\left(\frac{1}{a_5}\right)}{\log\left(r_{\max}\right)} := a_6 < \infty.$$

The final two inequalities follow immediately from the definitions of k and l, Corollary 6.6 and the observation that for $j = l$ and k we have

$$r_{\min} \exp\left(S_j\phi\left(\omega|j\right)\right) \leq \exp\left(S_{j+1}\phi\left(\omega|j+1\right)\right).$$

■

We now use this lemma to show that μ_u^q, $\mathcal{H}_{\mu_u}^{q,\beta(q)}$ and $\mathcal{P}_{\mu_u}^{q,\beta(q)}$ are equivalent measures.

Lemma 6.21 *There exists a positive and finite constant $\underline{K}$ such that for each $u \in V$ we have $\underline{K}\mu_u^q \leq \mathcal{H}_{\mu_u}^{q,\beta(q)} \llcorner K_u$.*

Proof: Let $E \subseteq K_u$ and $0 < \delta < \Delta$. Also let $\left(B_i\left(x_i, r_i\right)\right)_{i \in \mathbf{N}}$ be a centred δ-covering of E. For each $i \in \mathbf{N}$ choose $\omega_i \in E_u^{\mathbf{N}}$ such that $x_i = \pi_u\left(\omega_i\right)$ and $k_i, l_i \in \mathbf{N}$ such that

$$\text{diam}\left(K_{\omega_i|k_i+1}\right) \leq r_i < \text{diam}\left(K_{\omega_i|k_i}\right)$$

and

$$a_5 \text{ diam}\left(K_{\omega_i|l_i+1}\right) \leq r_i < a_5 \text{ diam}\left(K_{\omega_i|l_i}\right),$$

where a_5 is the constant appearing in Lemma 6.19. Then, applying Lemma 6.20, for each $i \in \mathbf{N}$ we have,

$$K_{\omega_i|k_i+1} \subseteq B\left(x_i, r_i\right), \tag{19}$$

$$K_u \cap B\left(x_i, r_i\right) \subseteq K_{\omega_i|l_i+1} \tag{20}$$

and there exists a finite constant a_6 independent of i such that $k_i - l_i \leq a_6$ for each i. Also there exist a positive and finite constant a_8 not depending on the i such that for each i

$$a_8^{-1} \exp\left(S_{l_i}\phi\left(\omega_i|l_i\right)\right) \leq r_i < a_8 \exp\left(S_{l_i}\phi\left(\omega_i|l_i\right)\right). \tag{21}$$

Now by using the two sided estimates in Equation 21 we can find $C_1 \in (0, \infty)$ such that

$$\left(\exp\left(S_{|l_i|}\psi\left(\omega_i|l_i\right)\right)\right)^{\beta(q)} \leq C_1\left(2r_i\right)^{\beta(q)}.$$

Also using Equations 19 and 20 together with the existence of a_6 we may find $C_2 \in (0, \infty)$ such that

$$p_{\omega_i|l_i}^q = \mu_u\left(K_{\omega_i|l_i}\right)^q \leq C_2\mu_u\left(B\left(x_i, r_i\right)\right)^q.$$

Now,

$$\begin{aligned}
\mu_u^q(E) &\leq \sum_i \mu_u^q\left(B\left(x_i, r_i\right)\right) \\
&\leq \sum_i \mu_u^q\left(K_{\omega_i|l_i}\right) \\
&\leq \sum_i A_q p_{\omega_i|l_i}^q \left(\exp\left(S_{|l_i|}\psi\left(\omega_i|l_i\right)\right)\right)^{\beta(q)} \\
&\leq A_q C_1 C_2 \sum_i \left(\mu_u\left(B\left(x_i, r_i\right)\right)\right)^q \left(2r_i\right)^{\beta(q)} \\
&:= \frac{1}{\underline{K}} \sum_i \left(\mu_u\left(B\left(x_i, r_i\right)\right)\right)^q \left(2r_i\right)^{\beta(q)}.
\end{aligned}$$

Thus taking infima over centred δ-coverings of E we have

$$\underline{K}\mu_u^q(E) \leq \mathcal{H}_{\mu_u,\delta}^{q,\beta(q)}(E) \leq \mathcal{H}_{\mu_u,0}^{q,\beta(q)}(E) \leq \mathcal{H}_{\mu_u}^{q,\beta(q)}(E).$$

■

Lemma 6.22 *There exists a positive and finite constant $\overline{K}$ such that for each $u \in V$, we have $\mathcal{P}^{q,\beta(q)}_{\mu_u} \leq \overline{K}\mu^q_u \llcorner K_u$.*

Proof: We begin by proving the following lemma:

Lemma 6.23 *There exists a positive and finite constant $\overline{K}$ such that for each $u \in V$ and $\tau \in E^{(*)}_u$ we have $\mathcal{P}^{q,\beta(q)}_{\mu_u}(K_\tau) \leq \overline{K}\mu^q_u(K_\tau)$.*

Proof: Let $\tau \in E^{(*)}_u$ and $\eta > 0$ be given. Since μ^q_u is finite and therefore outer regular we may choose an open and bounded set G_η such that $K_\tau \subseteq G_\eta$ and $\mu^q_u(G_\eta \backslash K_\tau) \leq \eta$. Clearly $\delta_\eta := \text{dist}(K_\tau, \mathbf{R}^n \backslash G_\eta) > 0$. Let $0 < \delta < \delta_\eta$ and $(B(x_i, r_i))_{i \in \mathbf{N}}$ be a centred δ-packing of K_τ. For each $i \in \mathbf{N}$ choose $\omega_i \in [\tau]$ such that $\pi_u(\omega_i) = x_i$ and integers k_i and l_i such that

$$\text{diam}\left(K_{\omega_i|k_i+1}\right) \leq r_i < \text{diam}\left(K_{\omega_i|k_i}\right)$$

and

$$a_5 \text{ diam}\left(K_{\omega_i|l_i+1}\right) \leq r_i < a_5 \text{ diam}\left(K_{\omega_i|l_i}\right).$$

Then, applying Lemma 6.20, for each $i \in \mathbf{N}$ we have,

$$K_{\omega_i|k_i+1} \subseteq B(x_i, r_i), \tag{22}$$

$$K_u \cap B(x_i, r_i) \subseteq K_{\omega_i|l_i+1} \tag{23}$$

and there exists a finite constant a_6 independent of i such that $k_i - l_i \leq a_6$ for each i. Also there exist a positive and finite constant a_7 not depending on the i such that for each i

$$a_7^{-1} \exp(S_{k_i}\phi(\omega_i|k_i)) \leq r_i < a_7 \exp(S_{k_i}\phi(\omega_i|k_i)). \tag{24}$$

Now, by using the two sided estimates in Equation 24, we can find $C_1 \in (0, \infty)$ such that

$$(2r_i)^{\beta(q)} \leq C_1 (\exp(S_{k_i}\psi(\omega_i|k_i)))^{\beta(q)}.$$

Also, using Equations 22 and 23 together with the existence of a_6, we may find $C_2 \in (0, \infty)$ such that

$$\mu_u(B(x_i, r_i))^q \leq C_2 \mu_u\left(K_{\omega_i|k_i}\right)^q.$$

Thus,

$$\begin{aligned}
\sum_i \mu_u(B(x_i, r_i))^q (2r_i)^{\beta(q)} &\leq C_1 C_2 \sum_i \mu_u\left(K_{\omega_i|k_i}\right)^q (\exp(S_{k_i}\psi(\omega_i|k_i)))^{\beta(q)} \\
&\leq \overline{K} \sum_i \mu^q_u\left(K_{\omega_i|k_i}\right) \\
&\leq \overline{K} \sum_i \mu^q_u(B(x_i, r_i)) \\
&= \overline{K}\mu^q_u\left(\bigcup_i B(x_i, r_i)\right) \\
&\leq \overline{K}\mu^q_u(G_\eta) \\
&\leq \overline{K}(\mu^q_u(K_\tau) + \eta)
\end{aligned}$$

Now, by taking suprema over δ-packings, we have that

$$\mathcal{P}^{q,\beta(q)}_{\mu_u,\delta}(K_\tau) \leq \overline{K}(\mu^q_u(K_\tau) + \eta)$$

for $\eta > 0$ and $0 < \delta < \delta_\eta$. Letting first $\delta \searrow 0$ and then $\eta \searrow 0$ we obtain

$$\mathcal{P}^{q,\beta(q)}_{\mu_u}(K_\tau) \leq \mathcal{P}^{q,\beta(q)}_{\mu_u,0}(K_\tau) \leq \overline{K}\left(\mu^q_u(K_\tau)\right).$$

■

Let $G \subseteq K_u$ be a relatively open subset of K_u and set $A = \left\{\tau \in E^{(*)}_u \middle| K_\tau \subseteq G\right\}$. By simple containment arguments and the fact that G is open we have that there exists a subset A_0 of A such that

$$G = \bigcup_{\tau \in A_0} K_\tau$$

and $[\tau] \cap [\alpha] = \emptyset$ for all $\tau, \alpha \in A_0$. Hence,

$$\begin{aligned}
\mathcal{P}^{q,\beta(q)}_{\mu_u}(G) &= \mathcal{P}^{q,\beta(q)}_{\mu_u}\left(\bigcup_{\tau \in A_0} K_\tau\right) \\
&\leq \sum_{\tau \in A_0} \mathcal{P}^{q,\beta(q)}_{\mu_u}(K_\tau) \\
&= \sum_{\tau \in A_0} \mathcal{P}^{q,\beta(q)}_{\mu_u}(K_\tau) \\
&\leq \overline{K} \sum_{\tau \in A_0} \mu^q_u(K_\tau) \\
&= \overline{K} \sum_{\tau \in A_0} \hat{\mu}_q([\tau]) \\
&= \overline{K}\hat{\mu}_q\left(\bigcup_{\tau \in A_0} [\tau]\right) \\
&\leq \overline{K}\hat{\mu}_q\left(\pi_u^{-1}(G)\right) \\
&= \overline{K}\mu^q_u(G).
\end{aligned}$$

Now since this holds for all open subsets of K_u and $\mathcal{P}^{q,\beta(q)}_{\mu_u}$ and μ^q_u are finite Borel measures we have the result. ■

If we recall that $\underline{a} = \inf_{q \in \mathbf{R}}\{\alpha(q)\}$ and $\overline{a} = \sup_{q \in \mathbf{R}}\{\alpha(q)\}$, then it is interesting to note that the results in Section 4.3 give us the following theorem as a corollary to Lemmas 6.21 and 6.22.

Theorem 6.24 *For each $u \in V$ and $q \in \mathbf{R}$, we have:*

1. $0 < \mathcal{H}^{q,\beta(q)}_{\mu_u}\left(K_{u,\alpha(q)}\right) \leq \mathcal{P}^{q,\beta(q)}_{\mu_u}\left(K_{u,\alpha(q)}\right) < \infty$;
2. $\mathrm{b}_{\mu_u}(q) = \mathrm{B}_{\mu_u}(q) = \beta(q)$;
3. $\mathrm{f}_{\mu_u}(q) = \mathrm{F}_{\mu_u}(q) = \beta^*(q)$ *for* $q \in (\underline{a}, \overline{a})$.

6.5 Open Questions

These calculations concerning the multifractal structure of graph directed self-conformal measures leave us with several interesting open questions.

The first group of questions relate to weaker separation conditions:

Question 1 *We have shown that if the GCIFS that we are considering satisfies the strong separation condition then for each $u \in V$:*

$$b_{\mu_u}(q) = B_{\mu_u}(q) = \beta(q)$$

and

$$0 < \mathcal{H}^{q,\beta(q)}_{\mu_u}\left(K_{u,\alpha(q)}\right) \leq \mathcal{P}^{q,\beta(q)}_{\mu_u}\left(K_{u,\alpha(q)}\right) < \infty.$$

Do these equations still hold for weaker separation conditions such as the strong open set condition or the weak separation condition introduced by Lau and Ngai in [LN1]?

We note that Das has addressed this question in the self-similar case (see [Das1]) and we also note that it would be more natural to consider the open set condition rather than the strong open set condition in Question 1. Unfortunately, however, it is unknown whether the two are equivalent in the self-conformal case. This leads us naturally to the following question:

Question 2 *It has been shown by Schief [Sc94] that in the self-similar case the strong open set condition is equivalent to the open set condition. This result was extend by Wang [Wan97] to the graph directed self-similar case. Is this also true in the self-conformal case?*

Question 3 *We have shown that if the GCIFS that we are considering satisfies the strong open set condition then for each $u \in V$ there exists an interval $(\underline{a}_u, \overline{a}_u)$ such that for $\alpha \in (\underline{a}_u, \overline{a}_u)$,*

$$f_{\mu_u}(\alpha) = F_{\mu_u}(\alpha) = \beta^*(\alpha).$$

Does the multifractal spectrum of the measures μ_u still coincide with the Legendre transform of β if the GCIFS satisfies weaker separation conditions than the strong open set condition e.g. the weak separation condition introduced by Lau and Ngai in [LN1]?

Our final question relates to the relationship between $\mathcal{H}^{q,\beta(q)}_{\mu_u}$ and $\mathcal{P}^{q,\beta(q)}_{\mu_u}$.

Question 4 *Given a strongly connected Mauldin-Williams graph satisfying the strong separation condition Olsen (see [Ol95]) has shown that for each $q \in \mathbf{R}$ there exists $c_q \in (0, \infty)$ such that*

$$\mathcal{H}^{q,\beta_u(q)}_{\mu_u} \llcorner \operatorname{supp}\mu_u = c_q\ \mathcal{P}^{q,\beta_u(q)}_{\mu_u} \llcorner \operatorname{supp}\mu_u.$$

Are the measures $\mathcal{H}^{q,\beta_u(q)}_{\mu_u}$ and $\mathcal{P}^{q,\beta_u(q)}_{\mu_u}$ also proportional in the strongly connected self-conformal case?

We suspect that it is not the case that $\mathcal{H}^{q,\beta_u(q)}_{\mu_u}$ and $\mathcal{P}^{q,\beta_u(q)}_{\mu_u}$ are proportional in the strongly connected self-conformal case. We do however make the following conjecture.

Conjecture 1 *Let $\mathcal{H}^{q,\beta_u(q)}_{\mu_u}$ and $\mathcal{P}^{q,\beta_u(q)}_{\mu_u}$ be the measures associated with a GCIFS satisfying the strong separation condition. There exists $c_q \in (0, \infty)$ such that*

$$c_q^{-1}\ \mathcal{P}^{q,\beta_u(q)}_{\mu_u} \llcorner \operatorname{supp}\mu_u \leq \mathcal{H}^{q,\beta_u(q)}_{\mu_u} \llcorner \operatorname{supp}\mu_u \leq c_q\ \mathcal{P}^{q,\beta_u(q)}_{\mu_u} \llcorner \operatorname{supp}\mu_u.$$

7 Relative Multifractal Analysis

In 1965, Billingsley published his book *Ergodic Theory and Information.* In this book Billingsley applies methods from ergodic theory to calculate the size of sets

$$K(\gamma) = \left\{ x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \;\middle|\; \lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} = \gamma \right\},$$

where μ, ν are probability measures on a metric space X. In [Caj81], Cajar also studies these sets in the code space. Anyone familiar with multifractal analysis will recognise this as a form of multifractal analysis. In several recent papers on multifractal analysis this type of multifractal analysis has re-emerged as mathematicians and physicists have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure, as opposed to Lebesgue measure (see [RS1], [LV98] and [Das98]). In this chapter we formalise these ideas by introducing a formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the 'multifractal formalism' as first introduced by Halsey et. al. [HJKPS86] and closely follows Olsen's formal treatment of this formalism in [Ol95].

7.1 Preliminaries

We begin our analysis by introducing two well known measures; the centred ν-Hausdorff measure and the ν-packing measure. For $q \in \mathbf{R}$, recall from Chapter 4, that $\varphi_q:[0,\infty) \to [0,\infty]$ is defined by

$$\begin{aligned} \varphi_q(x) &= \begin{cases} \infty & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } q < 0; \\ \varphi_q(x) &= 1 && \text{for } q = 0; \\ \varphi_q(x) &= \begin{cases} 0 & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} && \text{for } 0 < q. \end{aligned}$$

Given $\nu \in \mathcal{M}^1(X)$, for $s, \delta > 0$, set

$$\mathcal{H}^s_{\nu,\delta}(E) = \inf\{ \sum_i \varphi_s(\nu(B(x_i,r_i))) \mid (B(x_i,r_i))_i \text{ is a centred } \delta\text{-covering of } E \};\; E \neq \emptyset$$

$$\mathcal{H}^s_{\nu,\delta}(\emptyset) = 0; \qquad \mathcal{H}^s_{\nu,0}(E) = \sup_{\delta>0} \mathcal{H}^s_{\nu,\delta}(E); \qquad \mathcal{H}^s_{\nu}(E) = \sup_{F\subseteq E} \mathcal{H}^s_{\nu,0}(F);$$

$$\mathcal{P}^s_{\nu,\delta}(E) = \sup\{ \sum_i \varphi_s(\nu(B(x_i,r_i))) \mid (B(x_i,r_i))_i \text{ is a centred } \delta\text{-packing of } E \};\; E \neq \emptyset$$

$$\mathcal{P}^s_{\nu,\delta}(\emptyset) = 0; \qquad \mathcal{P}^s_{\nu,0}(E) = \inf_{\delta>0} \mathcal{P}^s_{\nu,\delta}(E); \qquad \mathcal{P}^s_{\nu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \mathcal{P}^s_{\nu,0}(E_i).$$

It is well known that these set functions are metric outer measures and that these measures define two dimension functions in the usual way, we denote them by $\dim_\nu$ and Dim_ν, respectively. These measures and dimensions are obviously related to the measures and dimensions introduced by Billingsley; the difference between the two is that Billingsley used centred ν-δ-coverings rather than centred δ-coverings.

7.2 Relative Multifractal Measures

We start our formalism by defining two generalised measures.

Definition 7.1 *Let X be a metric space. For $\mu, \nu \in \mathcal{M}^1(X)$, $E \subseteq X$, q, $t \in \mathbf{R}$ and $\delta > 0$ we make the following definitions:*

$$\mathcal{H}^{q,t}_{\mu,\nu,\delta}(E) = \inf\{ \sum_i \varphi_q(\mu(B(x_i,r_i)))\, \varphi_t(\nu(B(x_i,r_i))) \mid (B(x_i,r_i))_i \textit{ is a centred } \delta\textit{-covering of } E\}\;\; E \neq \emptyset;$$

$$\mathcal{H}^{q,t}_{\mu,\nu,\delta}(\emptyset)=0; \qquad \mathcal{H}^{q,t}_{\mu,\nu,0}(E)=\sup_{\delta>0}\mathcal{H}^{q,t}_{\mu,\nu,\delta}(E); \qquad \mathcal{H}^{q,t}_{\mu,\nu}(E)=\sup_{F\subseteq E}\mathcal{H}^{q,t}_{\mu,\nu,0}(F);$$

$$\mathcal{P}^{q,t}_{\mu,\nu,\delta}(E)=\sup\Big\{\sum_i \varphi_q\left(\mu\left(B\left(x_i,r_i\right)\right)\right)\varphi_t\left(\nu\left(B\left(x_i,r_i\right)\right)\right)\ \Big|\ \left(B\left(x_i,r_i\right)\right)_i \text{ is a centred } \delta\text{-packing of } E\Big\}\ \ E\neq\emptyset;$$

$$\mathcal{P}^{q,t}_{\mu,\nu,\delta}(\emptyset)=0; \qquad \mathcal{P}^{q,t}_{\mu,\nu,0}(E)=\inf_{\delta>0}\mathcal{P}^{q,t}_{\mu,\nu,\delta}(E); \qquad \mathcal{P}^{q,t}_{\mu,\nu}(E)=\inf_{E\subseteq\bigcup_i E_i}\sum_i \mathcal{P}^{q,t}_{\mu,\nu,0}(E_i),$$

where we set $0\cdot\infty=\infty\cdot 0=0$.

Proposition 7.2

1. *The set function* $\mathcal{H}^{q,t}_{\mu,\nu}$ *is a metric outer measure and thus a measure on the Borel algebra.*

2. *The set function* $\mathcal{P}^{q,t}_{\mu,\nu}$ *is a metric outer measure and thus a measure on the Borel algebra.*

Proof: Follows by standard arguments (see, for example, Propositions 2.2. and 2.3 in [Ol95]). ∎

Proposition 7.3 *There exist unique extended real valued numbers* $\dim^q_{\mu,\nu}(E)\in[-\infty,\infty]$, $\mathrm{Dim}^q_{\mu,\nu}(E)\in[-\infty,\infty]$, *and* $\Delta^q_{\mu,\nu}(E)\in[-\infty,\infty]$ *such that:*

$$\begin{aligned}
\mathcal{H}^{q,t}_{\mu,\nu}(E) &= \begin{cases}\infty & t<\dim^q_{\mu,\nu}(E)\\ 0 & \dim^q_{\mu,\nu}(E)<t;\end{cases}\\
\mathcal{P}^{q,t}_{\mu,\nu}(E) &= \begin{cases}\infty & t<\mathrm{Dim}^q_{\mu,\nu}(E)\\ 0 & \mathrm{Dim}^q_{\mu,\nu}(E)<t;\end{cases}\\
\mathcal{P}^{q,t}_{\mu,\nu,0}(E) &= \begin{cases}\infty & t<\Delta^q_{\mu,\nu}(E)\\ 0 & \Delta^q_{\mu,\nu}(E)<t.\end{cases}
\end{aligned}$$

Proof: This follows by elementary arguments. ∎

It is easily seen that for $t\geq 0$

$$\mathcal{H}^{0,t}_{\mu,\nu}=\mathcal{H}^t_\nu\ ,\ \mathcal{P}^{0,t}_{\mu,\nu}=\mathcal{P}^t_\nu\ ,\ \mathcal{P}^{0,t}_{\mu,\nu,0}=\mathcal{P}^t_{\nu,0}$$

where $\mathcal{H}^t_\nu$, $\mathcal{P}^t_\nu$ and $\mathcal{P}^t_{\nu,0}$ denote centred ν-Hausdorff t-measure, ν-packing t-measure and ν-pre-packing t-measure respectively. Thus if we denote ν-pre-packing dimension by Δ_ν then for $E\subseteq \operatorname{supp}\mu\cap\operatorname{supp}\nu$ we have

$$\dim_\nu(E)=\dim^0_{\mu,\nu}(E)\ ,\ \mathrm{Dim}_\nu(E)=\mathrm{Dim}^0_{\mu,\nu}(E),\ \text{ and } \Delta_\nu(E)=\Delta^0_{\mu,\nu}(E).$$

The following theorem summarises some of the important properties of these measures and dimension functions.

Theorem 7.4 *Let* $\mu,\nu\in\mathcal{M}^1\left(\mathbf{R}^d\right)$ *and* $q,t\in\mathbf{R}$. *Then*

1. $\mathcal{P}^{q,t}_{\mu,\nu}\leq\mathcal{P}^{q,t}_{\mu,\nu,0}$;

2. *there exists an integer* $\zeta\in\mathbf{N}$, *such that* $\mathcal{H}^{q,t}_{\mu,\nu}\leq\zeta\mathcal{P}^{q,t}_{\mu,\nu}$;

3. $\dim^q_{\mu,\nu}\leq\mathrm{Dim}^q_{\mu,\nu}\leq\Delta^q_{\mu,\nu}$;

4. *for* $\mu,\nu\in\mathcal{M}^1_D\left(\mathbf{R}^d\right)$, $\mathcal{H}^{q,t}_{\mu,\nu}\leq\mathcal{P}^{q,t}_{\mu,\nu}$.

Note: In fact, for $q \leq 0$ the condition in (4) that $\mu \in \mathcal{M}_D^1\left(\mathbf{R}^d\right)$ can be relaxed; similarly, for $t \leq 0$, the condition that $\nu \in \mathcal{M}_D^1\left(\mathbf{R}^d\right)$ can be relaxed.

Proof: (1) Follows immediately from the definitions.

(2) Let ζ be the integer that appears in the Besicovitch covering theorem. We start by proving that

$$\mathcal{H}_{\mu,\nu,0}^{q,t}(F) \leq \zeta \mathcal{P}_{\mu,\nu,0}^{q,t}(F) \tag{25}$$

for all $F \subseteq \mathbf{R}^d$. Let $\delta > 0$ and set $\mathcal{V} = \{B(x,r) \mid x \in F \text{ and } 0 < r < \delta\}$. It follows from the Besicovitch covering theorem that there exist ζ countable subfamilies $\left(B(x_{ij}, r_{ij})\right)_j$, $i = 1, \ldots, \zeta$ of $\mathcal{V}$ such that $\left(B(x_{ij}, r_{ij})\right)_{i,j}$ is a centred δ-covering of F and $\left(B(x_{ij}, r_{ij})\right)_j$ is a centred δ-packing of F for each i. Hence,

$$\mathcal{H}_{\mu,\nu,0}^{q,t}(F) \leq \sum_{i=1}^{\zeta} \sum_j \mu\left(B(x_{ij}, r_{ij})\right)^q \nu\left(B(x_{ij}, r_{ij})\right)^t \leq \sum_{i=1}^{\zeta} \mathcal{P}_{\mu,\nu,\delta}^{q,t}(F) \leq \zeta \mathcal{P}_{\mu,\nu,\delta}^{q,t}(F).$$

Letting $\delta \searrow 0$ yields Equation 25. Let $E \subseteq \mathbf{R}^d$ and $E \subseteq \bigcup_i E_i$. Then Equation 25 implies that

$$\begin{aligned}\mathcal{H}_{\mu,\nu}^{q,t}(E) = \mathcal{H}_{\mu,\nu}^{q,t}\left(\bigcup_i (E \cap E_i)\right) &\leq \sum_i \mathcal{H}_{\mu,\nu}^{q,t}(E \cap E_i) = \sum_i \sup_{F \subseteq E \cap E_i} \mathcal{H}_{\mu,\nu,0}^{q,t}(F) \leq \zeta \sum_i \sup_{F \subseteq E \cap E_i} \mathcal{P}_{\mu,\nu,0}^{q,t}(F) \\ &\leq \zeta \sum_i \mathcal{P}_{\mu,\nu,0}^{q,t}(E_i).\end{aligned}$$

Hence $\mathcal{H}_{\mu,\nu}^{q,t}(E) \leq \zeta \mathcal{P}_{\mu,\nu}^{q,t}(E)$.

(3) Follows immediately from (1) and (2).

(4) Let $E \subseteq \mathbf{R}^d$. For $m \in \mathbf{N}$ set

$$E_m = \left\{ x \in E \,\middle|\, \frac{\mu(B(x,5r))}{\mu(B(x,r))} < m \text{ and } \frac{\nu(B(x,5r))}{\nu(B(x,r))} < m \text{ for } 0 < r < \frac{1}{m} \right\},$$

where we set $\frac{a}{0} = 1$ for $a \geq 0$. Fix $m \in \mathbf{N}$ and let $F \subseteq E_m$. We start by showing that

$$\mathcal{H}_{\mu,\nu,0}^{q,t}(F) \leq \mathcal{P}_{\mu,\nu,0}^{q,t}(F).$$

First, let us assume that $\mathcal{P}_{\mu,\nu,0}^{q,t}(F) < \infty$ since if it is not there is nothing to prove. Let $\epsilon > 0$ and choose $\delta_1 > 0$ such that $\mathcal{H}_{\mu,\nu,0}^{q,t}(F) - \frac{\epsilon}{3} \leq \mathcal{H}_{\mu,\nu,\delta}^{q,t}(F)$ for $\delta \leq \delta_1$. Next choose $\delta_2 > 0$ such that $\mathcal{P}_{\mu,\nu,\delta}^{q,t}(F) \leq \mathcal{P}_{\mu,\nu,0}^{q,t}(F) + \frac{\epsilon}{3}$ for $\delta \leq \delta_2$. Let $\mathcal{V} = \{B(x,r) \mid x \in F \text{ and } r < \delta_1/5 \wedge \delta_2 \wedge 1/m\}$. Then $\mathcal{V}$ is a Vitali covering of F and thus we may use the Vitali Covering theorem to deduce that there exists a disjoint countable subfamily $\left(B_i := B(x_i, r_i)\right)_i \subseteq \mathcal{V}$ such that

$$F \setminus \bigcup_{i=1}^{k} B_i \subseteq \bigcup_{i=k+1}^{\infty} B(x_i, 5r_i) \qquad \text{for all } k.$$

Now, since $x_i \in E_m$ and $r_i \leq 1/m$,

$$\begin{aligned}\sum_i \mu\left(B(x_i, 5r_i)\right)^q \nu\left(B(x_i, 5r_i)\right)^t &\leq m^{q+t} \sum_i \mu\left(B(x_i, r_i)\right)^q \nu\left(B(x_i, r_i)\right)^t \leq m^{q+t} \mathcal{P}_{\mu,\nu,\delta_2}^{q,t}(F) \\ &\leq m^{q+t}\left(\mathcal{P}_{\mu,\nu,0}^{q,t} + \frac{\epsilon}{3}\right) < \infty.\end{aligned}$$

Hence, we may choose $K \in \mathbf{N}$ such that

$$\sum_{i=K+1}^{\infty} \mu\left(B(x_i, 5r_i)\right)^q \nu\left(B(x_i, 5r_i)\right)^t \leq \frac{\epsilon}{3}.$$

Thus,

$$\mathcal{H}^{q,t}_{\mu,\nu,0}(F) \le \mathcal{H}^{q,t}_{\mu,\nu,\delta_1}(F) + \frac{\epsilon}{3} \le \sum_{i=1}^{K} \mu(B_i)^q \nu(B_i)^t + \sum_{i=K+1}^{\infty} \mu(B(x_i,5r_i))^q \nu(B(x_i,5r_i))^t + \frac{\epsilon}{3}$$
$$\le \sum_i \mu(B_i)^q \nu(B_i)^t + \frac{\epsilon}{3} + \frac{\epsilon}{3} \le \mathcal{P}^{q,t}_{\mu,\nu,\delta_2}(F) + \frac{2\epsilon}{3} \le \mathcal{P}^{q,t}_{\mu,\nu,0}(F) + \epsilon$$

for all $\epsilon > 0$, and

$$\mathcal{H}^{q,t}_{\mu,\nu,0}(F) \le \mathcal{P}^{q,t}_{\mu,\nu,0}(F) \qquad \text{for all } F \subseteq E_m.$$

Let $E_m \subseteq \bigcup_i F_i$ then,

$$\mathcal{H}^{q,t}_{\mu,\nu}(E_m) \le \sum_i \mathcal{H}^{q,t}_{\mu,\nu}(E_m \cap F_i) = \sum_i \sup_{F \subseteq E_m \cap F_i} \mathcal{H}^{q,t}_{\mu,\nu,0}(F) \le \sum_i \sup_{F \subseteq E_m \cap F_i} \mathcal{P}^{q,t}_{\mu,\nu,0}(F)$$
$$\le \sum_i \mathcal{P}^{q,t}_{\mu,\nu,0}(F_i),$$

hence $\mathcal{H}^{q,t}_{\mu,\nu}(E_m) \le \mathcal{P}^{q,t}_{\mu,\nu}(E_m)$ for all $m \in \mathbf{N}$. The result follows since $E_m \nearrow E$. ∎

7.3 Dimension Functions

Our next step is to define three multifractal dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu} : \mathbf{R} \to [-\infty, \infty]$ by setting

$$b_{\mu,\nu}(q) = \dim^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu),\ B_{\mu,\nu}(q) = \operatorname{Dim}^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu),$$
$$\text{and } \Lambda_{\mu,\nu}(q) = \Delta^q_{\mu,\nu}(\operatorname{supp}\mu \cap \operatorname{supp}\nu).$$

The next theorem shows that these functions have some of the properties which physicists ascribe to the τ function which appears in the 'multifractal formalism'.

Theorem 7.5 *Let X be a metric space and $\mu, \nu \in \mathcal{M}^1(X)$, then the following hold.*

1. $\mathcal{P}^{q,t}_{\mu,\nu,0} \ge \mathcal{P}^{p,t}_{\mu,\nu,0}$ *for* $q \le p$ *and* $\mathcal{P}^{q,s}_{\mu,\nu,0} \ge \mathcal{P}^{q,t}_{\mu,\nu,0}$ *for* $s \le t$.
2. $\Lambda_{\mu,\nu}$ *is decreasing and convex.*
3. *The map* $(q,t) \to \mathcal{P}^{q,t}_{\mu,\nu,0}$ *is logarithmically convex* i.e. *for all* $\alpha \in [0,1]$, $p,q,s,t \in \mathbf{R}$ *and* $E \subseteq X$,
$$\mathcal{P}^{\alpha p+(1-\alpha)q, \alpha t+(1-\alpha)s}_{\mu,\nu,0}(E) \le \left(\mathcal{P}^{p,t}_{\mu,\nu,0}(E)\right)^{\alpha} \left(\mathcal{P}^{q,s}_{\mu,\nu,0}(E)\right)^{1-\alpha}.$$
4. $\mathcal{P}^{q,t}_{\mu,\nu} \ge \mathcal{P}^{p,t}_{\mu,\nu}$ *for* $q \le p$ *and* $\mathcal{P}^{q,s}_{\mu,\nu} \ge \mathcal{P}^{q,t}_{\mu,\nu}$ *for* $s \le t$.
5. $B_{\mu,\nu}$ *is decreasing and convex.*
6. $\mathcal{H}^{q,t}_{\mu,\nu} \ge \mathcal{H}^{p,t}_{\mu,\nu}$ *for* $q \le p$ *and* $\mathcal{H}^{q,s}_{\mu,\nu} \ge \mathcal{H}^{q,t}_{\mu,\nu}$ *for* $s \le t$.
7. $b_{\mu,\nu}$ *is decreasing.*

Proof: Parts (1), (4) and (6) follow since $x \to a^x$ is decreasing for $0 < a \le 1$.
(2) Let $\epsilon, \delta > 0$, then for all centred $(\epsilon \wedge \delta)$-packings $(B_i := B(x_i, r_i))_i$ of $\operatorname{supp}\mu \cap \operatorname{supp}\nu$,

$$\sum_i \mu(B_i)^{\alpha p+(1-\alpha)q} \nu(B_i)^{\alpha t+(1-\alpha)s} = \sum_i \left(\mu(B_i)^p \nu(B_i)^t\right)^{\alpha} \left(\mu(B_i)^t \nu(B_i)^s\right)^{1-\alpha}$$
$$\le \left(\sum_i \mu(B_i)^p \nu(B_i)^t\right)^{\alpha} \left(\sum_i \nu(B_i)^q \nu(B_i)^s\right)^{1-\alpha}$$
$$\le \left(\mathcal{P}^{q,t}_{\mu,\nu,\epsilon}(\operatorname{supp}\mu \cap \operatorname{supp}\nu)\right)^{\alpha} \left(\mathcal{P}^{q,s}_{\mu,\nu,\delta}(\operatorname{supp}\mu \cap \operatorname{supp}\nu)\right)^{1-\alpha}.$$

Hence,

$$\begin{aligned}\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\nu,\mu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu) &\leq \mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\nu,\mu,\epsilon\wedge\delta}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\\ &\leq \left(\mathcal{P}^{q,t}_{\mu,\nu,\epsilon}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{\alpha}\left(\mathcal{P}^{q,s}_{\mu,\nu,\delta}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{1-\alpha},\end{aligned}$$

for all $\epsilon,\delta>0$. Whence,

$$\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s}_{\nu,\mu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\leq\left(\mathcal{P}^{q,t}_{\mu,\nu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{\alpha}\left(\mathcal{P}^{q,s}_{\mu,\nu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{1-\alpha}.$$

(3) Set $s=\Lambda^{p}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)$ and $t=\Lambda^{q}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)$ and let $\epsilon>0$. Then,

$$\begin{aligned}\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha s+(1-\alpha)t+\epsilon}_{\mu,\nu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)&\leq\left(\mathcal{P}^{p,s+\epsilon}_{\mu,\nu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{\alpha}\left(\mathcal{P}^{q,t+\epsilon}_{\mu,\nu,0}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\right)^{1-\alpha}\\&=0.0=0\end{aligned}$$

i.e.

$$\Lambda^{\alpha p+(1-\alpha)q}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)\leq\alpha\Lambda^{p}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)+(1-\alpha)\Lambda^{q}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)+\epsilon.$$

The result follows by letting $\epsilon\searrow 0$.

(5) Let us set $B=B_{\mu,\nu}$. Then it follows from part (4) that B is decreasing thus we only require to show that B is convex. Let $p,q\in\mathbf{R}$, $\alpha\in[0,1]$ and $\epsilon>0$ and set $t=B(p)$ and $s=B(q)$. Now by definition,

$$\mathcal{P}^{q,s+\epsilon}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)=0=\mathcal{P}^{p,t+\epsilon}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu),$$

thus we can choose coverings $(H_i)_{i\in\mathbf{N}}$ and $(K_i)_{i\in\mathbf{N}}$ of $\operatorname{supp}\mu\cap\operatorname{supp}\nu$ such that $\sum_i\mathcal{P}^{p,t+\epsilon}_{\mu,\nu,0}(H_i)\leq 1$ and $\sum_i\mathcal{P}^{q,s+\epsilon}_{\mu,\nu,0}(K_i)\leq 1$. Now, for $n\in\mathbf{N}$ set $E_n=\bigcup_{i,j=1}^{n}(H_i\cap K_j)$. Then for each $n\in\mathbf{N}$ it follows from (2), (3) and Hölder's inequality that,

$$\begin{aligned}\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha t+(1-\alpha)s+\epsilon}_{\mu,\nu}(E_n)&=\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha(t+\epsilon)+(1-\alpha)(s+\epsilon)}_{\mu,\nu}\left(\bigcup_{i,j=1}^{n}(H_i\cap K_j)\right)\\
&\leq\sum_{i,j=1}^{n}\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha(t+\epsilon)+(1-\alpha)(s+\epsilon)}_{\mu,\nu}(H_i\cap K_j)\\
&\leq\sum_{i,j=1}^{n}\mathcal{P}^{\alpha p+(1-\alpha)q,\alpha(t+\epsilon)+(1-\alpha)(s+\epsilon)}_{\mu,\nu,0}(H_i\cap K_j)\\
&\leq\sum_{i,j=1}^{n}\left(\mathcal{P}^{p,t+\epsilon}_{\mu,\nu,0}(H_i\cap K_j)\right)^{\alpha}\left(\mathcal{P}^{q,s+\epsilon}_{\mu,\nu,0}(H_i\cap K_j)\right)^{1-\alpha}\\
&\leq\left(\sum_{i,j=1}^{n}\mathcal{P}^{p,t+\epsilon}_{\mu,\nu,0}(H_i\cap K_j)\right)^{\alpha}\left(\sum_{i,j=1}^{n}\mathcal{P}^{q,s+\epsilon}_{\mu,\nu,0}(H_i\cap K_j)\right)^{1-\alpha}\\
&\leq\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{P}^{p,t+\epsilon}_{\mu,\nu,0}(H_i)\right)^{\alpha}\left(\sum_{j=1}^{n}\sum_{i=1}^{n}\mathcal{P}^{q,s+\epsilon}_{\mu,\nu,0}(K_j)\right)^{1-\alpha}\\
&\leq\left(n\sum_{i=1}^{n}\mathcal{P}^{p,t+\epsilon}_{\mu,\nu,0}(H_i)\right)^{\alpha}\left(n\sum_{j=1}^{n}\mathcal{P}^{q,s+\epsilon}_{\mu,\nu,0}(K_j)\right)^{1-\alpha}\\
&\leq n^{\alpha}n^{1-\alpha}=n<\infty.\end{aligned}$$

This gives us that $\mathrm{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(E_n) \le \alpha t + (1-\alpha)s + \epsilon$ for all $n \in \mathbf{N}$. Thus, since $(\operatorname{supp}\mu \cap \operatorname{supp}\nu) \subseteq \bigcup_n E_n$, we have

$$\begin{aligned} B(\alpha p + (1-\alpha)q) = {} & \mathrm{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(\operatorname{supp}\mu \cap \operatorname{supp}\nu) \le \mathrm{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}\left(\bigcup_n E_n\right) \\ & = \sup_n \mathrm{Dim}_{\mu,\nu}^{\alpha p+(1-\alpha)q}(E_n) \le \alpha B(p) + (1-\alpha)B(q) + \epsilon. \end{aligned}$$

The result follows since $\epsilon > 0$ is arbitrary.

(7) Follows immediately from (6). ∎

Corollary 7.6 *Let $\mu, \nu \in \mathcal{M}^1(X)$ be such that* $\operatorname{supp}\mu = \operatorname{supp}\nu$. *We have:*

1. *for $q < 1$, $0 \le b_{\mu,\nu}(q) \le B_{\mu,\nu}(q) \le \Lambda_{\mu,\nu}(q)$;*
2. *$b_{\mu,\nu}(1) = B_{\mu,\nu}(1) = \Lambda_{\mu,\nu}(1) = 0$;*
3. *for $q > 1$, $b_{\mu,\nu}(q) \le B_{\mu,\nu}(q) \le \Lambda_{\mu,\nu}(q) \le 0$.*

Proof: This follows immediately from the above theorem and definitions. ∎

7.4 The ν-Multifractal Spectrum

Having defined the generalised multifractal ν-Hausdorff and ν-packing measures and the ν-Hausdorff, ν-packing and ν-pre-packing dimension functions we wish to demonstrate their usefulness by showing their connection to the ν-multifractal spectra.

Given $\mu, \nu \in \mathcal{M}^1(X)$ the *upper* respectively *lower local dimension of μ with respect to ν at $x \in X$* is defined by,

$$\overline{\gamma}_{\mu,\nu}(x) = \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} \qquad \text{respectively} \qquad \underline{\gamma}_{\mu,\nu}(x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))}.$$

If $\overline{\gamma}_{\mu,\nu}(x) = \underline{\gamma}_{\mu,\nu}(x)$ then the common value, known as the *local dimension of μ with respect to ν at x*, is denoted by $\gamma_{\mu,\nu}(x)$. Given $\mu, \nu \in \mathcal{M}^1(X)$, for $\gamma \ge 0$, set

$$\begin{aligned} \overline{K}^{\gamma} = {} & \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \overline{\gamma}_{\mu,\nu}(x) \le \gamma\right\}; \qquad \overline{K}_{\gamma} = \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma \le \overline{\gamma}_{\mu,\nu}(x)\right\}; \\ \underline{K}^{\gamma} = {} & \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \underline{\gamma}_{\mu,\nu}(x) \le \gamma\right\}; \qquad \underline{K}_{\gamma} = \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma \le \underline{\gamma}_{\mu,\nu}(x)\right\}. \end{aligned}$$

Also, let

$$K(\gamma) = \underline{K}_{\gamma} \cap \overline{K}^{\gamma} = \{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma_{\mu,\nu}(x) = \gamma\}.$$

Finally, the *ν-Hausdorff multifractal spectrum of μ* and the *ν-packing multifractal spectrum of μ* are defined in the following way,

$$g_{\mu,\nu}(\gamma) = \dim_{\nu} K(\gamma) \qquad \text{and} \qquad G_{\mu,\nu}(\gamma) = \mathrm{Dim}_{\nu} K(\gamma).$$

With these definitions we have the following theorem.

Theorem 7.7 *Let X be a metric space and $\mu, \nu \in \mathcal{M}^1(X)$. Also fix $\gamma \ge 0$, $q, t \in \mathbf{R}$ and $\delta > 0$ such that $\delta \le \gamma q + t$. Then we have the following:*

1. *(a) $\mathcal{H}_{\nu}^{\gamma q+t+\delta}\left(\overline{K}^{\gamma}\right) \le \mathcal{H}_{\mu,\nu}^{q,t}\left(\overline{K}^{\gamma}\right)$ for $0 \le q$.*

   (b) $\mathcal{H}_{\nu}^{\gamma q+t+\delta}\left(\underline{K}_{\gamma}\right) \le \mathcal{H}_{\mu,\nu}^{q,t}\left(\underline{K}_{\gamma}\right)$ for $q \le 0$.

(c) *If* $0 \leq \gamma q + b_{\mu,\nu}(q)$ *then*

$$\begin{array}{llll} \dim_\nu\left(\overline{K}^\gamma\right) \leq & \gamma q + b_{\mu,\nu}(q) & \text{for } 0 \leq q \\ \dim_\nu\left(\underline{K}_\gamma\right) \leq & \gamma q + b_{\mu,\nu}(q) & \text{for } q \leq 0. \end{array}$$

In particular, $\dim_\nu\left(\overline{K}^\gamma\right) \leq \gamma$.

2. (a) $\mathcal{P}_\nu^{\gamma q+t+\delta}\left(\overline{K}^\gamma\right) \leq \mathcal{P}_{\mu,\nu}^{q,t}\left(\overline{K}^\gamma\right)$ *for* $0 \leq q$.

   (b) $\mathcal{P}_\nu^{\gamma q+t+\delta}\left(\underline{K}_\gamma\right) \leq \mathcal{P}_{\mu,\nu}^{q,t}\left(\underline{K}_\gamma\right)$ *for* $q \leq 0$.

   (c) *If* $0 \leq \gamma q + B_{\mu,\nu}(q)$ *then*

$$\begin{array}{llll} \mathrm{Dim}_\nu\left(\overline{K}^\gamma\right) \leq & \gamma q + B_{\mu,\nu}(q) & \text{for } 0 \leq q \\ \mathrm{Dim}_\nu\left(\underline{K}_\gamma\right) \leq & \gamma q + B_{\mu,\nu}(q) & \text{for } q \leq 0. \end{array}$$

   In particular, $\mathrm{Dim}_\nu\left(\overline{K}^\gamma\right) \leq \gamma$.

3. (a) *If* $A \subseteq \overline{K}^\gamma$ *is Borel then* $\mathcal{H}_{\mu,\nu}^{q,t}(A) \leq \mathcal{H}_\nu^{\gamma q+t-\delta}(A)$ *for* $q \leq 0$.

   (b) *If* $A \subseteq \underline{K}_\gamma$ *is Borel then* $\mathcal{H}_{\mu,\nu}^{q,t}(A) \leq \mathcal{H}_\nu^{\gamma q+t-\delta}(A)$ *for* $0 \leq q$. *In particular, if* $A \subseteq \underline{K}_\gamma$ *is Borel and* $\mu(A) > 0$ *then* $\gamma \leq \dim_\nu(A)$.

4. (a) *If* $A \subseteq \overline{K}^\gamma$ *is Borel then* $\mathcal{P}_{\mu,\nu}^{q,t}(A) \leq \mathcal{P}_\nu^{\gamma q+t-\delta}(A)$ *for* $q \leq 0$.

   (b) *If* $A \subseteq \underline{K}_\gamma$ *is Borel then* $\mathcal{P}_{\mu,\nu}^{q,t}(A) \leq \mathcal{P}_\nu^{\gamma q+t-\delta}(A)$ *for* $0 \leq q$. *In particular, if* $A \subseteq \underline{K}_\gamma$ *is Borel and* $\mu(A) > 0$ *then* $\gamma \leq \mathrm{Dim}_\nu(A)$.

Proof: An exhaustive proof of this theorem would require considerable repetition. All of the ideas needed to prove this theorem can be found in the proofs of Propositions 2.5 through 2.8 in [Ol95]. To aid the reader in using these ideas we prove $1(b)$ and $4(a)$.

$1(b)$ It is well known that the statement is true for $q = 0$. For $m \in \mathbf{N}$ let us set

$$T_m = \left\{ x \in \underline{K}_\gamma \middle| \gamma + \frac{\delta}{q} \leq \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Given $m \in \mathbf{N}$ let us choose η such that $0 < \eta < \frac{1}{m}$ and let $(B_i := B(x_i, r_i))_i$ be a centred η-covering of $E \subseteq T_m$. Then we have,

$$\begin{array}{lll} \frac{\log(\mu(B(x_i,r_i)))}{\log(\nu(B(x_i,r_i)))} & \geq & \gamma + \frac{\delta}{q} \quad \Rightarrow \\ \mu(B(x_i,r_i)) & \leq & \nu(B(x_i,r_i))^{\gamma+\frac{\delta}{q}} \quad \Rightarrow \\ \mu(B(x_i,r_i))^q & \geq & \nu(B(x_i,r_i))^{\gamma q+\delta} \quad \Rightarrow \\ \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t & \geq & \nu(B(x_i,r_i))^{\gamma q+\delta+t}. \end{array}$$

Hence,

$$\mathcal{H}_{\nu,\eta}^{\gamma q+\delta+t}(E) \leq \sum_i \nu(B(x_i,r_i))^{\gamma q+\delta+t} \leq \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t.$$

Now from this we can deduce that for $\eta < \frac{1}{m}$, $\mathcal{H}_{\nu,\eta}^{\gamma q+\delta+t}(E) \leq \mathcal{H}_{\mu,\nu,\eta}^{q,t}(E)$ and letting $\eta \searrow 0$ gives that for all $E \subseteq T_m$,

$$\mathcal{H}_{\nu,0}^{\gamma q+\delta+t}(E) \leq \mathcal{H}_{\mu,\nu,0}^{q,t}(E) \leq \mathcal{H}_{\mu,\nu}^{q,t}(E) \leq \mathcal{H}_{\mu,\nu}^{q,t}(T_m).$$

Hence,

$$\mathcal{H}_\nu^{\gamma q+\delta+t}(T_m) \leq \mathcal{H}_{\mu,\nu}^{q,t}(T_m).$$

The result follows since $T_m \nearrow \underline{K}_\gamma$.

4(a) Once again for $q = 0$ the statement is well known. For $m \in \mathbf{N}$ let us set,

$$T_m = \left\{ x \in A \middle| \frac{\log\left(\mu\left(B\left(x,r\right)\right)\right)}{\log\left(\nu\left(B\left(x,r\right)\right)\right)} \leq \gamma - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Now given $m \in \mathbf{N}$, $E \subseteq T_m$ and $0 < \eta < \frac{1}{m}$ let $\left(B\left(x_i, r_i\right)\right)_{i \in \mathbf{N}}$ be a centred δ-packing of E. Then we have,

$$\begin{array}{lll} \frac{\log(\mu(B(x_i,r_i)))}{\log(\nu(B(x_i,r_i)))} & \leq & \gamma - \frac{\delta}{q} \;\; \Rightarrow \\ \mu\left(B\left(x_i,r_i\right)\right) & \geq & \nu\left(B\left(x_i,r_i\right)\right)^{\gamma - \frac{\delta}{q}} \;\; \Rightarrow \\ \mu\left(B\left(x_i,r_i\right)\right)^q & \leq & \nu\left(B\left(x_i,r_i\right)\right)^{\gamma q - \delta} \;\; \Rightarrow \\ \mu\left(B\left(x_i,r_i\right)\right)^q \nu\left(B\left(x_i,r_i\right)\right)^t & \leq & \nu\left(B\left(x_i,r_i\right)\right)^{\gamma q + t - \delta}. \end{array}$$

Hence,

$$\sum_i \mu\left(B\left(x_i,r_i\right)\right)^q \nu\left(B\left(x_i,r_i\right)\right)^t \leq \sum_i \nu\left(B\left(x_i,r_i\right)\right)^{\gamma q + t - \delta} \leq \mathcal{P}^{\gamma q + t - \delta}_{\nu,\eta}\left(E\right).$$

From this we can deduce that for $\eta < \frac{1}{m}$, $\mathcal{P}^{q,t}_{\mu,\nu,\eta}\left(E\right) \leq \mathcal{P}^{\gamma q + t - \delta}_{\nu,\eta}\left(E\right)$. Thus letting $\eta \searrow 0$ gives that for all $E \subseteq T_m$,

$$\mathcal{P}^{q,t}_{\mu,\nu,0}\left(E\right) \leq \mathcal{P}^{\gamma q + t - \delta}_{\nu,0}\left(E\right).$$

Finally, let $\left(E_i\right)_{i \in \mathbf{N}}$ be a covering of T_m. Then we have,

$$\begin{aligned} \mathcal{P}^{q,t}_{\mu,\nu}\left(T_m\right) &\leq \mathcal{P}^{q,t}_{\mu,\nu}\left(\bigcup_i \left(T_m \cap E_i\right)\right) \leq \sum_i \mathcal{P}^{q,t}_{\mu,\nu}\left(T_m \cap E_i\right) \leq \sum_i \mathcal{P}^{q,t}_{\mu,\nu,0}\left(T_m \cap E_i\right) \leq \sum_i \mathcal{P}^{\gamma q + t - \delta}_{\nu,0}\left(T_m \cap E_i\right) \\ &\leq \sum_i \mathcal{P}^{\gamma q + t - \delta}_{\nu,0}\left(E_i\right). \end{aligned}$$

Hence,

$$\mathcal{P}^{q,t}_{\mu,\nu}\left(T_m\right) \leq \mathcal{P}^{\gamma q + t - \delta}_{\nu}\left(T_m\right),$$

and the result follows since $A = \bigcup_m T_m$. ∎

Theorem 7.7 allows us to consider the relationship between the dimension functions $b_{\mu,\nu}$ and $B_{\mu,\nu}$ and the ν-multifractal spectra. For $\mu, \nu \in \mathcal{M}^1\left(X\right)$ set,

$$\underline{a}_{\mu,\nu} := \sup_{0<q} -\frac{b_{\mu,\nu}\left(q\right)}{q}, \;\; \overline{a}_{\mu,\nu} := \inf_{q<0} -\frac{b_{\mu,\nu}\left(q\right)}{q}, \;\; \underline{A}_{\mu,\nu} := \sup_{0<q} -\frac{B_{\mu,\nu}\left(q\right)}{q}, \;\; \text{and } \overline{A}_{\mu,\nu} := \inf_{q<0} -\frac{B_{\mu,\nu}\left(q\right)}{q}$$

then

$$\underline{A}_{\mu,\nu} \leq \underline{a}_{\mu,\nu} \text{ and } \overline{a}_{\mu,\nu} \leq \overline{A}_{\mu,\nu}.$$

With these definitions we have the following theorem.

Theorem 7.8 *Let X be a metric space, $\mu, \nu \in \mathcal{M}^1\left(X\right)$ and $\gamma \geq 0$. Then the following hold:*

1. $\underline{a}_{\mu,\nu} \leq \inf \overline{\gamma}_{\mu,\nu}\left(x\right) \leq \sup \overline{\gamma}_{\mu,\nu}\left(x\right) \leq \overline{A}_{\mu,\nu}$ *and* $\underline{A}_{\mu,\nu} \leq \inf \underline{\gamma}_{\mu,\nu}\left(x\right) \leq \sup \underline{\gamma}_{\mu,\nu}\left(x\right) \leq \overline{a}_{\mu,\nu}$;

2.

$$g_{\mu,\nu}\left(\gamma\right) = \left\{ \begin{array}{lll} \leq & b^*_{\mu,\nu}\left(\gamma\right) & \gamma \in \left(\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}\right) \\ = & 0 & \gamma \in [0,\infty) \setminus [\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}]; \end{array} \right.$$

3.

$$G_{\mu,\nu}(\gamma) = \begin{cases} \leq & B^*_{\mu,\nu}(\gamma) \quad \gamma \in (\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}) \\ = & 0 \quad\quad\quad\quad \gamma \in [0,\infty) \setminus [\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}]. \end{cases}$$

Proof: This theorem follows immediately from Theorem 7.7 and the following lemma. ∎

Lemma 7.9 *If X is a metric space, $\mu, \nu \in \mathcal{M}^1(X)$ and $\gamma \geq 0$, then*

1. $\underline{K}^\gamma = \emptyset$ *for* $\gamma < \underline{A}_{\mu,\nu}$.
2. $\underline{K}_\gamma = \emptyset$ *for* $\overline{a}_{\mu,\nu} < \gamma$.
3. $\overline{K}_\gamma = \emptyset$ *for* $\overline{A}_{\mu,\nu} < \gamma$.
4. $\overline{K}^\gamma = \emptyset$ *for* $\gamma < \underline{a}_{\mu,\nu}$.

Proof: (1) Suppose that $\gamma < \underline{A}_{\mu,\nu}$ and $x \in \underline{K}^\gamma$, then there exist real numbers $\epsilon > 0$ and $q_0 > 0$ such that $\gamma + \epsilon < -\frac{B_{\mu,\nu}(q_0)}{q_0}$ *i.e.* $-q_0(\gamma+\epsilon) > B_{\mu,\nu}(q_0)$. Now set $t = -q_0(\gamma+\epsilon)$, then since $x \in \underline{K}^\gamma$,

$$\liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))} \leq \gamma < \gamma + \epsilon.$$

We can thus choose a sequence $(r_n)_{n\in\mathbf{N}}$ such that $r_n \to 0$, $0 < r_n < \frac{1}{n}$ and $\frac{\log(\mu(B(x,r_n)))}{\log(\nu(B(x,r_n)))} < \gamma + \epsilon$. These conditions imply that,

$$\mu(B(x,r_n))^{q_0} (\nu(B(x,r_n)))^t \geq (\nu(B(x,r_n)))^{q_0(\gamma+\epsilon)+t} = 1.$$

Hence for all $n \in \mathbf{N}$ we have

$$\mathcal{P}^{q_0,t}_{\mu,\nu,\frac{1}{n}}(\{x\}) \geq \mu(B(x,r_n))^{q_0} (\nu(B(x,r_n)))^t \geq 1$$

which clearly implies that $\mathcal{P}^{q_0,t}_{\mu,\nu}(\{x\}) \geq 1$. This gives that $-q_0(\gamma+\epsilon) = t \leq \mathrm{Dim}^{q_0}_{\mu,\nu}(\{x\}) \leq B_{\mu,\nu}(q_0)$ which contradicts the fact that $-q_0(\gamma+\epsilon) > B_{\mu,\nu}(q_0)$.

(2) Suppose that $\overline{a}_{\mu,\nu} < \gamma$ and $x \in \underline{K}_\gamma$, then there exist real numbers $\epsilon > 0$ and $q_0 < 0$ such that $\gamma - \epsilon > -\frac{b_{\mu,\nu}(q_0)}{q_0}$, *i.e.* $-q_0(\gamma-\epsilon) > b_{\mu,\nu}(q_0)$. Now set $t = -q_0(\gamma-\epsilon)$, then since $x \in \underline{K}_\gamma$,

$$\gamma + \epsilon < \gamma \leq \liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))}.$$

We can thus choose $r_0 > 0$ such that for $0 < r < r_0$ we have $\gamma - \epsilon < \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))}$. This condition implies that for all r such that $0 < r < r_0$,

$$\mu(B(x,r))^{q_0} (\mu(B(x,r)))^t \geq (\nu(B(x,r)))^{q_0(\gamma-\epsilon)+t} = 1.$$

Hence we can argue that $\mathcal{H}^{q_0,t}_{\mu,\nu}(\{x\}) \geq 1$, this in turn implies that $-q_0(\gamma-\epsilon) = t \leq \dim^{q_0}_{\mu,\nu}(\{x\}) \leq b_{\mu,\nu}(q_0)$ which contradicts the fact that $-q_0(\gamma-\epsilon) > b_{\mu,\nu}(q_0)$.

(3) and (4) follow in a similar way. ∎

Theorem 7.10 *Let X be a metric space and $\mu, \nu \in \mathcal{M}^1(X)$. If $A \subseteq K(\gamma)$ is a Borel set such that $\mathcal{H}^{q,t}_{\mu,\nu}(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\gamma q + t \geq 0$. Then,*

$$\dim_\nu(A) \geq \gamma q + t.$$

In particular, if $b_{\mu,\nu}$ is differentiable at q and we set $\gamma(q) = -b'_{\mu,\nu}(q)$ then provided that $b^*_{\mu,\nu}(\gamma(q)) \geq 0$ and $\mathcal{H}^{q,b_{\mu,\nu}(q)}_{\mu,\nu}(K(\gamma(q))) > 0$ we have

$$g_{\mu,\nu}(\gamma(q)) = b^*_{\mu,\nu}(\gamma(q)).$$

Proof: Follows immediately from Theorem 7.7 ∎

Theorem 7.11 *Let X be a metric space and $\mu, \nu \in \mathcal{M}^1(X)$. If $A \subseteq K(\gamma)$ is a Borel set such that $\mathcal{P}_{\mu,\nu}^{q,t}(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\gamma q + t \geq 0$. Then,*

$$\mathrm{Dim}_\nu(A) \geq \gamma q + t.$$

In particular, if $B_{\mu,\nu}$ is differentiable at q and we set $\gamma(q) = -B'_{\mu,\nu}(q)$ then provided that $B^*_{\mu,\nu}(\gamma(q)) \geq 0$ and $\mathcal{P}_\mu^{q,B_{\mu,\nu}(q)}(K(\gamma(q))) > 0$ we have

$$G_{\mu,\nu}(\gamma(q)) = B^*_{\mu,\nu}(\gamma(q)).$$

Proof: Follows immediately from Theorem 7.7 ∎

7.5 The Multifractal Spectrum

While the ν-multifractal spectra are of theoretical interest it is more natural for us to want to know what the actual multifractal spectra are *i.e.* to know the Hausdorff and packing dimensions of the sets $K(\gamma)$. In the general case the best that we can do is to decompose the sets $K(\gamma)$ according to the ν-local dimension of their points and then calculate the size of the subset of $K(\gamma)$ whose points have ν local dimension α.

Given $\mu, \nu \in \mathcal{M}^1(X)$ the *upper* respectively *lower local dimension of ν at $x \in X$* is defined by,

$$\overline{\alpha}_\nu(x) = \limsup_{r \searrow 0} \frac{\log \nu(B(x,r))}{\log r} \qquad \text{respectively} \qquad \underline{\alpha}_\nu(x) = \liminf_{r \searrow 0} \frac{\log \nu(B(x,r))}{\log r}.$$

If $\overline{\alpha}_\nu(x) = \underline{\alpha}_\nu(x)$ then the common value, known as the *local dimension of ν at x*, is denoted by $\alpha_\nu(x)$. Given $\mu, \nu \in \mathcal{M}^1(X)$ for $\gamma, \alpha \geq 0$ set

$$\begin{aligned}
K^{+,+}(\gamma,\alpha) = &\ \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \overline{\gamma}_{\mu,\nu}(x) \leq \gamma \text{ and } \overline{\alpha}_\nu(x) \leq \alpha\right\}; \\
K^{-,+}(\gamma,\alpha) = &\ \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma \leq \underline{\gamma}_{\mu,\nu}(x) \text{ and } \overline{\alpha}_\nu(x) \leq \alpha\right\}; \\
K^{+,-}(\gamma,\alpha) = &\ \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \overline{\gamma}_{\mu,\nu}(x) \leq \gamma \text{ and } \alpha \leq \underline{\alpha}_\nu(x)\right\}; \\
K^{-,-}(\gamma,\alpha) = &\ \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma \leq \underline{\gamma}_{\mu,\nu}(x) \text{ and } \alpha \leq \underline{\alpha}_\nu(x)\right\}.
\end{aligned}$$

Also, let

$$K(\gamma,\alpha) = \{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu \mid \gamma_{\mu,\nu}(x) = \gamma \text{ and } \alpha_\nu(x) = \alpha\}.$$

Finally, set

$$f_{\mu,\nu}(\gamma,\alpha) = \dim_{\mathrm{H}} K(\gamma,\alpha) \qquad \text{and} \qquad F_{\mu,\nu}(\gamma,\alpha) = \dim_{\mathrm{P}} K(\gamma,\alpha).$$

Theorem 7.12 *Let X be a metric space and $\mu, \nu \in \mathcal{M}^1(X)$. Also fix $\gamma, \alpha \geq 0$, $q, t \in \mathbf{R}$ and $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq \gamma q + t$ and $\delta_2 \leq \alpha(\gamma q + t - \delta_1)$. Then we have the following:*

1. *(a) $\mathcal{H}^{\alpha(\gamma q+t+\delta_1)+\delta_2}(K^{+,+}(\gamma,\alpha)) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}_{\mu,\nu}^{q,t}(K^{+,+}(\gamma,\alpha))$ for $0 \leq q$.*

   (b) $\mathcal{H}^{\alpha(\gamma q+t+\delta_1)+\delta_2}(K^{-,+}(\gamma,\alpha)) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}_{\mu,\nu}^{q,t}(K^{-,+}(\gamma,\alpha))$ for $q \leq 0$.

2. *(a) $\mathcal{P}^{\alpha(\gamma q+t+\delta_1)+\delta_2}(K^{+,+}(\gamma,\alpha)) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{P}_{\mu,\nu}^{q,t}(K^{+,+}(\gamma,\alpha))$ for $0 \leq q$.*

   (b) $\mathcal{P}^{\alpha(\gamma q+t+\delta_1)+\delta_2}(K^{-,+}(\gamma,\alpha)) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{P}_{\mu,\nu}^{q,t}(K^{-,+}(\gamma,\alpha))$ for $q \leq 0$.

3. *(a) If $A \subseteq K^{+,-}(\gamma,\alpha)$ is Borel then, for $q \leq 0$, $\mathcal{H}_{\mu,\nu}^{q,t}(A) \leq 2^{-(\alpha(\gamma q+t-\delta_1)-\delta_2)}\mathcal{H}^{\alpha(\gamma q+t-\delta_1)-\delta_2}(A)$.*

   (b) If $A \subseteq K^{-,-}(\gamma,\alpha)$ is Borel then, for $0 \leq q$, $\mathcal{H}_{\mu,\nu}^{q,t}(A) \leq 2^{-(\alpha(\gamma q+t-\delta_1)-\delta_2)}\mathcal{H}^{\alpha(\gamma q+t-\delta_1)-\delta_2}(A)$.

4. (a) *If $A \subseteq K^{+,-}(\gamma,\alpha)$ is Borel then, for $q \leq 0$, $\mathcal{P}^{q,t}_{\mu,\nu}(A) \leq 2^{-(\alpha(\gamma q+t-\delta_1)-\delta_2)}\mathcal{P}^{\alpha(\gamma q+t-\delta_1)-\delta_2}(A)$.*

(b) *If $A \subseteq K^{-,-}(\gamma,\alpha)$ is Borel then, for $0 \leq q$, $\mathcal{P}^{q,t}_{\mu,\nu}(A) \leq 2^{-(\alpha(\gamma q+t-\delta_1)-\delta_2)}\mathcal{P}^{\alpha(\gamma q+t-\delta_1)-\delta_2}(A)$.*

Proof: An exhaustive proof of this theorem would require considerable repetition. Thus we only prove $1(a)$ and $4(b)$. These two parts show how the technical details work for the Hausdorff and packing cases. All other details follow by choosing T_m correctly.

$1(a)$ It is well known that the statement is true for $q=0$. For $m \in \mathbf{N}$ let us set,

$$T_m = \left\{ x \in K^{+,+}(\gamma,\alpha) \,\middle|\, \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))} \leq \gamma + \frac{\delta_1}{q},\ \frac{\log(\nu(B(x,r)))}{\log r} \leq \alpha + \frac{\delta_2}{\gamma q + t + \delta_1} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Given $m \in \mathbf{N}$ let us choose η such that $0 < \eta < \frac{1}{m}$ and let $(B_i := B(x_i,r_i))_i$ be a centred η-covering of $E \subseteq T_m$. Then we have,

$$\begin{array}{rcl} \frac{\log(\mu(B(x_i,r_i)))}{\log(\nu(B(x_i,r_i)))} & \leq & \gamma + \frac{\delta_1}{q} \;\Rightarrow \\ \mu(B(x_i,r_i)) & \geq & \nu(B(x_i,r_i))^{\gamma+\frac{\delta_1}{q}} \;\Rightarrow \\ \mu(B(x_i,r_i))^q & \geq & \nu(B(x_i,r_i))^{\gamma q+\delta_1} \;\Rightarrow \\ \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t & \geq & \nu(B(x_i,r_i))^{\gamma q+\delta_1+t}. \end{array}$$

Also, we have,

$$\begin{array}{rcl} \frac{\log(\nu(B(x,r)))}{\log r} & \leq & \alpha + \frac{\delta_2}{\gamma q+t+\delta_1} \;\Rightarrow \\ \nu(B(x_i,r_i)) & \geq & r_i^{\alpha+\frac{\delta_2}{\gamma q+t+\delta_1}} \;\Rightarrow \\ \nu(B(x_i,r_i))^{\gamma q+t+\delta_1} & \geq & r_i^{\alpha(\gamma q+\delta_1)+\delta_2}. \end{array}$$

Putting these together we have that

$$r_i^{\alpha(\gamma q+\delta_1)+\delta_2} \leq \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t.$$

Hence,

$$\mathcal{H}^{\alpha(\gamma q+\delta_1+t)+\delta_2}_\eta(E) \leq \sum_i (2r_i)^{\alpha(\gamma q+\delta_1)+\delta_2} \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2} \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t.$$

Now from this we can deduce that for $\eta < \frac{1}{m}$, $\mathcal{H}^{\alpha(\gamma q+\delta_1)+\delta_2}_\eta(E) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}^{q,t}_{\mu,\nu,\eta}(E)$ and letting $\eta \searrow 0$ gives that for all $E \subseteq T_m$,

$$\mathcal{H}^{\alpha(\gamma q+\delta_1)+\delta_2}_0(E) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}^{q,t}_{\mu,\nu,0}(E) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}^{q,t}_{\mu,\nu}(E) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}^{q,t}_{\mu,\nu}(T_m).$$

Hence,

$$\mathcal{H}^{\alpha(\gamma q+\delta_1)+\delta_2}(T_m) \leq 2^{\alpha(\gamma q+\delta_1)+\delta_2}\mathcal{H}^{q,t}_{\mu,\nu}(T_m).$$

The result follows since $T_m \nearrow K^{+,+}(\gamma,\alpha)$.

$4(b)$ Once again for $q=0$ the statement is well known. For $m \in \mathbf{N}$ let us set,

$$T_m = \left\{ x \in A \,\middle|\, \gamma - \frac{\delta_1}{q} \leq \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))} \text{ and } \alpha - \frac{\delta_2}{\gamma q + t - \delta_2} \leq \frac{\log(\nu(B(x,r)))}{\log r} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Now given $m \in \mathbf{N}$, $E \subseteq T_m$ and $0 < \eta < \frac{1}{m}$ let $(B(x_i,r_i))_{i\in\mathbf{N}}$ be a centred δ-packing of E. Then we have,

$$\begin{array}{rcl} \frac{\log(\mu(B(x_i,r_i)))}{\log(\nu(B(x_i,r_i)))} & \geq & \gamma - \frac{\delta_1}{q} \;\Rightarrow \\ \mu(B(x_i,r_i)) & \leq & \nu(B(x_i,r_i))^{\gamma-\frac{\delta_1}{q}} \;\Rightarrow \\ \mu(B(x_i,r_i))^q & \leq & \nu(B(x_i,r_i))^{\gamma q-\delta_1} \;\Rightarrow \\ \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t & \leq & \nu(B(x_i,r_i))^{\gamma q+t-\delta_1}. \end{array}$$

Also, we have,

$$\begin{array}{rcl} \frac{\log(\nu(B(x,r)))}{\log r} & \geq & \alpha - \frac{\delta_2}{\gamma q + t - \delta_1} \Rightarrow \\ \nu(B(x_i, r_i)) & \leq & r_i^{\alpha - \frac{\delta_2}{\gamma q + t - \delta_1}} \Rightarrow \\ \nu(B(x_i, r_i))^{\gamma q + t - \delta_1} & \leq & r_i^{\alpha(\gamma q + t - \delta_1) - \delta_2}. \end{array}$$

Putting these together we have that

$$\mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \leq r_i^{\alpha(\gamma q + t - \delta_1) - \delta_2}.$$

Hence,

$$\begin{aligned} \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t &\leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \sum_i (2r_i)^{\alpha(\gamma q + t - \delta_1) - \delta_2} \\ &\leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \mathcal{P}_\eta^{\alpha(\gamma q + t - \delta_1) - \delta_2}(E). \end{aligned}$$

From this we can deduce that for $\eta < \frac{1}{m}$, $\mathcal{P}_{\mu,\nu,\eta}^{q,t}(E) \leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \mathcal{P}_\eta^{\alpha(\gamma q + t - \delta_1) - \delta_2}(E)$. Thus letting $\eta \searrow 0$ gives that for all $E \subseteq T_m$,

$$\mathcal{P}_{\mu,\nu,0}^{q,t}(E) \leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \mathcal{P}_0^{-\alpha(\gamma q + t - \delta_1) - \delta_2}(E).$$

Finally, let $(E_i)_{i \in \mathbf{N}}$ be a covering of T_m. Then we have,

$$\begin{aligned} \mathcal{P}_{\mu,\nu}^{q,t}(T_m) &\leq \mathcal{P}_{\mu,\nu}^{q,t}\left(\bigcup_i (T_m \cap E_i)\right) \leq \sum_i \mathcal{P}_{\mu,\nu}^{q,t}(T_m \cap E_i) \leq \sum_i \mathcal{P}_{\mu,\nu,0}^{q,t}(T_m \cap E_i) \\ &\leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \sum_i \mathcal{P}_0^{\alpha(\gamma q + t - \delta_1) - \delta_2}(T_m \cap E_i) \leq 2^{-(\alpha(\gamma q + t - \delta_1) - \delta_2)} \sum_i \mathcal{P}_0^{\alpha(\gamma q + t - \delta_1) - \delta_2}(E_i). \end{aligned}$$

Hence,

$$\mathcal{P}_{\mu,\nu}^{q,t}(T_m) \leq 2^{-\alpha(\gamma q + t - \delta_1) - \delta_2} \mathcal{P}^{\alpha(\gamma q + t - \delta_1) - \delta_2}(T_m),$$

and the result follows since $A = \bigcup_m T_m$. ∎

Theorem 7.12 allows us to consider the relationship between the dimension functions $b_{\mu,\nu}$ and $B_{\mu,\nu}$ and the spectra $f_{\mu,\nu}$ and $F_{\mu,\nu}$. We start by giving an upper bound theorem.

Theorem 7.13 *Let X be a metric space, $\mu, \nu \in \mathcal{M}^1(X)$ and $\gamma, \alpha \geq 0$. Then the following hold:*

1.

$$f_{\mu,\nu}(\gamma, \alpha) = \begin{cases} \leq & \alpha \cdot b_{\mu,\nu}^*(\gamma) \quad \gamma \in (\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}) \\ = & 0 \qquad\qquad\quad \gamma \in [0, \infty) \setminus [\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}]; \end{cases}$$

2.

$$F_{\mu,\nu}(\gamma, \alpha) = \begin{cases} \leq & \alpha \cdot B_{\mu,\nu}^*(\gamma) \quad \gamma \in (\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}) \\ = & 0 \qquad\qquad\quad \gamma \in [0, \infty) \setminus [\underline{a}_{\mu,\nu}, \overline{a}_{\mu,\nu}]. \end{cases}$$

Proof: Follows immediately from Theorem 7.12 and Lemma 7.9. ∎

Theorem 7.14 *Let X be a metric space, $\gamma, \alpha \geq 0$ and $\mu, \nu \in \mathcal{M}^1(X)$. If $A \subseteq K(\gamma, \alpha)$ is a Borel set such that $\mathcal{H}_{\mu,\nu}^{q,t}(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\gamma q + t \geq 0$. Then,*

$$\dim_{\mathrm{H}}(A) \geq \alpha(\gamma q + t).$$

In particular, if $b_{\mu,\nu}$ is differentiable at q and we set $\gamma(q) = -b'_{\mu,\nu}(q)$ then provided that $b_{\mu,\nu}^*(\gamma(q)) \geq 0$ and $\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(K(\gamma(q), \alpha)) > 0$ we have

$$f_{\mu,\nu}(\gamma(q), \alpha) = \alpha \cdot b_{\mu,\nu}^*(\gamma(q)).$$

Proof: Follows immediately from Theorem 7.12 ■

Theorem 7.15 *Let X be a metric space, $\gamma, \alpha \geq 0$ and $\mu, \nu \in \mathcal{M}^1(X)$. If $A \subseteq K(\gamma, \alpha)$ is a Borel set such that $\mathcal{P}^{q,t}_{\mu,\nu}(A) > 0$, where $q, t \in \mathbf{R}$ are such that $\gamma q + t \geq 0$. Then,*

$$\dim_{\mathrm{P}}(A) \geq \alpha(\gamma q + t).$$

In particular, if $B_{\mu,\nu}$ is differentiable at q and we set $\gamma(q) = -B'_{\mu,\nu}(q)$ then provided that $B^*_{\mu,\nu}(\gamma(q)) \geq 0$ and $\mathcal{P}^{q, B_{\mu,\nu}(q)}_{\mu}(K(\gamma(q), \alpha)) > 0$ we have

$$F_{\mu,\nu}(\gamma(q), \alpha) = \alpha \cdot B^*_{\mu,\nu}(\gamma(q)).$$

Proof: Follows immediately from Theorem 7.12 ■

8 The Relative Multifractal Spectrum of Graph Directed Self-Conformal Measures

In this chapter we give an example to illustrate the general theory which we have been developing for relative multifractal analysis. We investigate the relative multifractal structure of one graph-directed self-conformal measure with respect to another in the case where the two measures are based on the same iterated function scheme.

In this chapter we let $G = \left(V, E, (T_e)_{e\in E}, (p_e)_{e\in E}\right)$ and $G' = \left(V, E, (T_e)_{e\in E}, (m_e)_{e\in E}\right)$ be two GCIFSs with probabilities based on the same GCIFS and satisfying the SOSC. We adopt the notation given in the Chapter 6 for G. For G', we let ν_u, $\hat{\nu}_u$, ν_u^q, $\hat{\nu}_q$, $m_{\min}$, $m_{\max}$ and m_τ play the respective roles of μ_u, $\hat{\mu}_u$, μ_u^q, $\hat{\mu}_q$, $p_{\min}$, $p_{\max}$ and p_τ. In addition we set $\chi(\omega) = \log m_{\omega_1}$.

Our investigation of the relative multifractal structure of the measures μ_u with respect to ν_u is performed in three stages. First we calculate the spectra $g_{\mu_u,\nu_u}(\gamma)$ and $G_{\mu_u,\nu_u}(\gamma)$ and then the spectra $f_{\mu_u,\nu_u}(\gamma,\alpha)$ and $F_{\mu_u,\nu_u}(\gamma,\alpha)$. Finally, we introduce functions α_{ν_u}, ζ_{μ_u,ν_u} and γ_{μ_u,ν_u} such that

$$\dim_{\mathrm{H}}\left(K_u\left(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)\right)\right) = \dim_{\mathrm{P}}\left(K_u\left(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)\right)\right) = \alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right).$$

In addition we give a counter example to the natural conjecture that,

$$\dim_{\mathrm{H}}\left(K_u\left(\gamma_{\mu_u,\nu_u}(q)\right)\right) = \dim_{\mathrm{P}}\left(K_u\left(\gamma_{\mu_u,\nu_u}(q)\right)\right) = \alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right).$$

8.1 The multifractal spectra $g_{\mu_u,\nu_u}(\gamma)$ and $G_{\mu_u,\nu_u}(\gamma)$

In order to analyse the sets

$$K_u(\gamma) = \left\{x \in \operatorname{supp}\mu_u \cap \operatorname{supp}\nu_u \,\middle|\, \lim_{r\searrow 0}\frac{\log\mu_u(B(x,r))}{\log\nu_u(B(x,r))} = \gamma\right\}$$

it is natural to consider the Gibbs state of the function $q\phi + \zeta\chi$. Using a suitable generalisation of Lemma 6.7 we are able to introduce a differentiable function ζ_{μ_u,ν_u} such that $P\left(q, \zeta_{\mu_u,\nu_u}(q)\right) = 0$, where $P(q,\zeta) = P(q\phi + \zeta\chi)$. This leads us to consider the Gibbs state of the function $q\phi + \zeta_{\mu_u,\nu_u}(q)\chi$, which we denote by $\hat{\rho}_q$. By definition $\hat{\rho}_q$ satisfies the following, there exists $c \in (0,\infty)$ such that for all $\tau \in E^{(*)}$,

$$c^{-1}p_\tau^q m_\tau^{\zeta_{\mu_u,\nu_u}(q)} \le \hat{\rho}_q([\tau]) \le c p_\tau^q m_\tau^{\zeta_{\mu_u,\nu_u}(q)}.$$

This being the case we hope that the reader will see that the calculation of the ν-multifractal spectra is equivalent to the calculation of the multifractal spectra of a graph directed self-similar measure. In particular, it is equivalent to calculating the multifractal spectra of the invariant measures of the GCIFS $H = \left(V, E, (S_e)_{e\in E}, (p_e)_{e\in E}\right)$, where the maps S_e are chosen to satisfy the following conditions; they have contraction ratio m_e and map $[0,1]$ into $[0,1]$ such that for all $e, f \in E_u$, $S_e([0,1]) \cap S_f([0,1])$ is either a singleton or empty. For in analysing this measures associated with H, one would introduce auxiliary measures $\hat{\mu}_q$ satisfying,

$$\hat{\mu}_q([\tau]) = p_\tau^q m_\tau^{\zeta(q)} \qquad \text{for all } \tau \in E^{(*)},$$

where $\zeta(q)$ is defined to be the number which makes the spectral radius of the matrix with elements $\sum_{e\in E_{u,v}} p_e^q m_e^{\zeta(q)}$ equal to 1, see [EM92].

The above considerations serve to justify the following theorem. Let $\gamma_{\mu_u,\nu_u}(q) = -\zeta'_{\mu_u,\nu_u}(q)$ and set $\underline{a} = \inf_{q\in\mathbf{R}}\{\gamma_{\mu_u,\nu_u}(q)\}$ and $\overline{a} = \sup_{q\in\mathbf{R}}\{\gamma_{\mu_u,\nu_u}(q)\}$.

Theorem 8.1 *If $\underline{a} < \overline{a}$ then for all $u \in V$,*

1. $\dim_\nu K_u(\gamma) = \operatorname{Dim}_\nu K_u(\gamma) = \zeta^*_{\mu_u,\nu_u}(\gamma)$ *for $\gamma \in (\underline{a}, \overline{a})$, where $\zeta^*_{\mu_u,\nu_u}$ denotes the Legendre transform of ζ_{μ_u,ν_u}.*

2. $K_u(\gamma) = \emptyset$ *for $\gamma \notin [\underline{a}, \overline{a}]$.*

8.2 The multifractal spectra $f_{\mu_u,\nu_u}(\gamma,\alpha)$ and $F_{\mu_u,\nu_u}(\gamma,\alpha)$

In this section we prove the main result of this example *i.e.* that for two pairs of graph directed self-conformal measures we have equality in Theorem 7.13.

Theorem 8.2 *If $\underline{a} < \overline{a}$ then for each $\gamma \in (\underline{a}, \overline{a})$ there exist $q \in \mathbf{R}$ such that $\gamma_{\mu_u,\nu_u}(q) = \gamma$ and we have that for all $u \in V$,*

1. $f_{\mu_u,\nu_u}(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)) = F_{\mu_u,\nu_u}(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)) = \alpha_{\nu_u}(q)\left(\zeta^*_{\mu_u,\nu_u}(\gamma_{\mu_u,\nu_u}(q))\right)$

2. $K_u(\gamma,\alpha) = \emptyset$ *for* $\gamma \notin [\underline{a}, \overline{a}]$.

Proof: (1) The upper bound follows from Theorem 7.13; the lower bound follows from Theorem 8.5 below.

(2) Follows from Lemma 7.9. ■

Our final task in this section is to introduce the material needed to prove Theorem 8.5 used in the above proof. The above section indicates that the measure $\hat{\rho}_q$ which satisfies

$$\hat{\rho}_q([\tau]) = p_\tau^q m_\tau^{\zeta_{\mu_u,\nu_u}(q)} \qquad \text{for all } \tau \in E^{(*)},$$

is important in this task. We note that we can choose c equal to one because we know by uniqueness that this Bernoulli measure must coincide with the Gibbs State. We also require ρ_u^q, its projection under π_u onto K_u. At this stage we note the following lemma which follows as a corollary to Lemma 6.11.

Lemma 8.3 *For $q \in \mathbf{R}$ and $u \in V$ we have $\int |\log \operatorname{dist}(x, \partial J_u)| d\rho_u^q(x) < \infty$.*

Set $\lambda(q) = \int \psi d\hat{\rho}_q$, $\eta(q) = \int \phi d\hat{\rho}_q$, $\kappa(q) = \int \chi d\hat{\rho}_q$, $\gamma_{\mu_u,\nu_u}(q) = \eta(q)/\kappa(q)$, $\theta_{\mu_u,\nu_u}(q) = \eta(q)/\lambda(q)$ and $\alpha_{\nu_u}(q) = \kappa(q)/\lambda(q)$. With these definitions in place let us note that $\theta_{\mu_u,\nu_u}(q) = \gamma_{\mu_u,\nu_u}(q)\alpha_{\nu_u}(q)$ and that the γ_{μ_u,ν_u} that we have defined here coincides with the γ_{μ_u,ν_u} that we defined in the last section (this follows by a similar argument to that given in Chapter 6 for α). Now the Ergodic theorem gives us the following lemma.

Lemma 8.4 *With $\lambda(q)$, $\eta(q)$ and $\kappa(q)$ defined as above:*

1. $\lim_{n\to\infty} \frac{1}{n} S_n\psi(\omega|n) = \lambda(q)$ *for* $\hat{\rho}_q$*-a.a.*$\omega \in E^{\mathbf{N}}$;

2. $\lim_{n\to\infty} \frac{1}{n} \log p_{\omega|n} = \eta(q)$ *for* $\hat{\rho}_q$*-a.a.*$\omega \in E^{\mathbf{N}}$;

3. $\lim_{n\to\infty} \frac{1}{n} \log m_{\omega|n} = \kappa(q)$ *for* $\hat{\rho}_q$*-a.a.*$\omega \in E^{\mathbf{N}}$.

We now use this lemma to show that ρ_u^q is a measure with local dimension almost surely equal to $\alpha_{\nu_u}(q)\zeta^*_{\mu_u,\nu_u}(\gamma_{\mu_u,\nu_u}(q))$ supported on $K_u(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)) := K_u \cap K(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q))$.

Recall that for $u \in V$ and $r \in (0,1)$,

$$\hat{\Gamma}_{u,r} = \left\{\tau \in E_u^{(*)} \mid \exp\left(S_{|\tau|}\psi(\tau)\right) < \frac{r}{a_1 \operatorname{diam} J_u} \le \exp\left(S_{|\tau|-1}\psi(\tau||\tau|-1)\right)\right\}.$$

Theorem 8.5 *Given $u \in V$ and $q \in \mathbf{R}$, for ρ_u^q-a.a.$x \in K_u$*

1.

$$\lim_{r\to 0} \frac{\log \mu_u(B(x,r))}{\log r} = \theta_{\mu_u,\nu_u}(q),$$

2.

$$\lim_{r\to 0} \frac{\log \nu_u(B(x,r))}{\log r} = \alpha_{\nu_u}(q),$$

3.

$$\lim_{r\to 0}\frac{\log \mu_u(B(x,r))}{\log \nu_u(B(x,r))}=\gamma_{\mu_u,\nu_u}(q),$$

4.

$$\lim_{r\to 0}\frac{\log \rho_u^q(B(x,r))}{\log r}=\alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q)+\zeta_{\mu_u,\nu_u}(q)\right).$$

5. Thus,

$$\dim_{\mathrm{H}} K_u\left(\gamma_{\mu_u,\nu_u}(q),\alpha_{\nu_u}(q)\right)\geq \alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q)+\zeta_{\mu_u,\nu_u}(q)\right).$$

Proof: (1) Given $r>0$ and $\omega\in E_u^{\mathbf{N}}$ choose $k_r(\omega)\in\mathbf{N}$ such that $\omega|k_r(\omega)\in\hat{\Gamma}_{u,r}$. Then $J_{\omega|k_r(\omega)}\subseteq B(\pi(\omega),r)$. Therefore with $k_r=k_r(\omega)$,

$$\begin{aligned}\frac{\log \mu_u(B(\pi(\omega),r))}{\log r}\leq\frac{\log\mu_u\left(J_{\omega|k_r}\right)}{\log r}&\leq\frac{\log p_{\omega|k_r}}{S_{k_r-1}\psi(\omega|k_r-1)+\log\operatorname{diam}J+\log a_1}\\&=\frac{\log p_{\omega|k_r}}{k_r}/\frac{S_{k_r-1}\psi(\omega|k_r-1)+\log\operatorname{diam}J+\log a_1}{k_r}\end{aligned}$$

and hence, $\limsup_{r\to 0}\frac{\log\mu_u(B(\pi(\omega),r))}{\log r}\leq\theta_{\mu_u,\nu_u}(q)$ for $\hat{\rho}_q$-a.a.$\omega\in E_u^{\mathbf{N}}$. Now since $\rho_u^q=\hat{\rho}_q\circ\pi_u^{-1}$ $\limsup_{r\to 0}\frac{\log\mu_u(B(x,r))}{\log r}\leq\theta_{\mu_u,\nu_u}(q)$ for ρ_u^q-a.a.$x\in K_u$.

To get the opposite inequality we define the following functions: for $u\in V$ and $m\in\mathbf{N}$, let $d_{u,m}\colon E_u^{\mathbf{N}}\to\mathbf{R}$ be given by, $d_{u,m}(\omega)=\operatorname{dist}\left(\pi_u(\omega),\partial J_{\omega|m}\right)$. Lemma 6.5 gives us that $d_{u,m}(\omega)\geq a_1^{-1}\exp\left(S_m\psi(\omega|m)\right)d_{i(\omega_m),0}(\sigma^m(\omega))$. For $m=0,1,\ldots$, let us set $\Sigma_m=\left\{\omega\in E^{\mathbf{N}}\mid d_{i(\omega),m}(\omega)>0\right\}$. It follows from Lemma 8.3 that $\hat{\rho}_q\left(E^{\mathbf{N}}\backslash\Sigma_m\right)=0$ for $m=0,1,\ldots$. Also if we set

$$\Sigma=\left\{\omega\in E^{\mathbf{N}}\mid d_{i(\omega),m}(\omega)>0,\qquad\text{for }m=0,1,\ldots\right\}$$

then $\hat{\rho}_q\left(E^{\mathbf{N}}\backslash\Sigma\right)=0$ since $\Sigma=\cap_m\Sigma_m$. Now for $0<r<1$ and $\omega\in\Sigma$ we are able to choose $m_r(\omega)\in\mathbf{N}$ such that $d_{u,m_r(\omega)+1}(\omega)\leq r<d_{u,m_r(\omega)}$. Then by the definition of $d_{u,m}$, $B(\pi_u(\omega),r)\subseteq J_{\omega|m_r(\omega)}$ thus with $m_r=m_r(\omega)$ we have

$$\begin{aligned}\frac{\log\mu_u(B(\pi(\omega),r))}{\log r}&\geq\frac{\log\mu_u\left(J_{\omega|m_r}\right)}{\log r}\\&\geq\frac{\log p_{\omega|m_r}}{S_{m_r+1}\psi(\omega|m_r+1)+\log d_{i(\omega_{m_r+1}),0}\left(\sigma^{m_r+1}(\omega)\right)-\log a_1}\\&=\frac{\log p_{\omega|m_r}}{m_r}/\frac{S_{m_r+1}\psi(\omega|m_r+1)+\log d_{i(\omega_{m_r+1}),0}\left(\sigma^{m_r+1}(\omega)\right)-\log a_1}{m_r}.\end{aligned}$$

Since for all $u\in V$, by Lemma 8.3, $\log d_{u,0}$ is integrable, the ergodic theorem gives us that,

$$\lim_{k\to\infty}\frac{\log d_{i(\omega_{k+1}),0}\left(\sigma^{k+1}(\omega)\right)}{k}=0.$$

Hence, $\liminf_{r\to 0}\frac{\log\mu_u(B(\pi(\omega),r))}{\log r}\geq\theta_{\mu_u,\nu_u}(q)$ for $\hat{\rho}_q$-a.a.$\omega\in E_u^{\mathbf{N}}$. Thus, $\liminf_{r\to 0}\frac{\log\mu_u(B(x,r))}{\log r}\geq\theta_{\mu_u,\nu_u}(q)$ for ρ_u^q-a.a.$x\in K_u$.

(2) This follows by the same argument as that used in (1) with μ_u replaced by ν_u and p replaced by m.

(3) Follows since

$$\frac{\log\mu_u(B(x,r))}{\log\nu_u(B(x,r))}=\frac{\frac{\log\mu_u(B(x,r))}{\log r}}{\frac{\log\nu_u(B(x,r))}{\log r}},$$

and $\gamma_{\mu_u,\nu_u}(q) = \theta_{\mu_u,\nu_u}(q)/\alpha_{\nu_u}(q)$.

(4) Follows by similar arguments to (1) if we use the following equality,

$$\hat{\rho}_q([\tau]) = p_\tau^q m_\tau^{\zeta_{\mu_u,\nu_u}(q)}.$$

(5) Follows immediately from (2), (3) and (4). ∎

8.3 An Example

Having calculated the Hausdorff and packing dimensions of the sets $K_u(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q))$ we would hope that we could use this calculation to calculate the Hausdorff and packing dimensions of $K_u(\gamma_{\mu_u,\nu_u}(q))$. A natural conjecture is that $\dim_H(K_u(\gamma_{\mu_u,\nu_u}(q))) = \dim_P(K_u(\gamma_{\mu_u,\nu_u}(q))) = \alpha_{\mu_u,\nu_u}(q)(q\,\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))$. We now give an example to show that there exist pairs of invariant measures based on the same GCIFS such that this is not the case.

Binomial measures are examples of measures which can be defined using GCIFSs. Let μ be the binomial measure supported on $[0,1]$ based on the probability vector $(p, 1-p)$, where $p \in \left(0, \frac{1}{2}\right)$ *i.e.* μ is the unique probability on $[0,1]$ satisfying $\mu = p \cdot \mu \circ T_1^{-1} + (1-p)\,\mu \circ T_2^{-1}$, where $T_1(x) = x/2$ and $T_2(x) = x/2 + \frac{1}{2}$. Set $\nu = \mu$. Then by definition we have that $\gamma_{\mu,\nu}(q) = 1$ for all q. This implies that $\zeta_{\mu,\nu}(q) = 1 - q$. Thus $\alpha_\nu(q)(q\gamma_{\mu,\nu}(q) + \zeta_{\mu,\nu}(q)) = \alpha_\nu(q)(q + 1 - q) = \alpha_\nu(q)$. Also, for all $x \in [0,1]$,

$$\lim_{r \searrow 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} = 1.$$

Thus for all q we have that $K(\gamma_{\mu_u,\nu_u}(q)) = [0,1]$. Hence $\dim_H(K(\gamma_{\mu,\nu}(q))) = \dim_H([0,1]) = 1$. Now, it is well known (see, for example, [CM92]) that $\alpha_\nu(q)$ is a decreasing function of q such that $\lim_{q\to\infty} \alpha_\nu(q) = \frac{\log(1-p)}{\log(1/2)} < 1$. Thus there exists q_0 such that for all $q > q_0$, $\alpha_\nu(q) < 1$. Hence there exists a q such that $\alpha_\nu(q) \neq 1$. This gives us a counter example to the conjecture that $\dim_H(K_u(\gamma_{\mu_u,\nu_u}(q))) = \dim_P(K_u(\gamma_{\mu_u,\nu_u}(q))) = \alpha_{\nu_u}(q)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))$ for all pairs of of invariant measures based on the same GCIFS.

8.4 Relative Multifractal Spectra

While the above example is a counter example to our initial conjecture we feel that it is more than likely that the situation where we set $\mu = \nu$ is a degenerate case. Our conjecture is as follows:

Conjecture 2

1. *If* $\nu_u \ll \mu_u$ *then there exist* q *such that* $\dim_H(K_u(\gamma_{\mu_u,\nu_u}(q))) \neq \alpha_{\nu_u}(q)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))$ *and there exist* q *such that* $\dim_P(K_u(\gamma_{\mu_u,\nu_u}(q))) \neq \alpha_{\nu_u}(q)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))$.

2. *If* $\mu_u \perp \nu_u$ *then* $\dim_H(K_u(\gamma_{\mu_u,\nu_u}(q))) = \dim_P(K_u(\gamma_{\mu_u,\nu_u}(q))) = \alpha_{\nu_u}(q)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))$ *for all* q.

For $q \in \mathbf{R}$ set

$$E(q) = \left\{ x \in \operatorname{supp}\mu_u \cap \operatorname{supp}\nu_u \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu_u(B(x,r))}{\log \nu_u(B(x,r))} = \gamma_{\mu_u,\nu_u}(q),\ \limsup_{r\searrow 0} \frac{\log \nu_u(B(x,r))}{\log r} < \alpha_{\nu_u}(q) \right\}$$

and

$$F(q) = \left\{ x \in \operatorname{supp}\mu_u \cap \operatorname{supp}\nu_u \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu_u(B(x,r))}{\log \nu_u(B(x,r))} = \gamma_{\mu_u,\nu_u}(q),\ \liminf_{r\searrow 0} \frac{\log \nu_u(B(x,r))}{\log r} > \alpha_{\nu_u}(q) \right\}.$$

Then

$$K_u(\gamma_{\mu_u,\nu_u}(q)) = K(\gamma_{\mu_u,\nu_u}(q), \alpha_{\nu_u}(q)) \bigcup E(q) \bigcup F(q).$$

So far what we have been able to prove is the following:

Theorem 8.6 *If the GCIFS that we are considering satisfies the strong separation condition, then for* $0 \le q \le 1$,

$$\dim_{\mathrm{H}}(E(q)) \le \dim_{\mathrm{P}}(E(q)) \le \alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right).$$

Proof: We start our proof by quoting the following lemma which can be proved by using standard arguments from the literature (see, for example, [Ol95]).

Lemma 8.7 *Let the GCIFS that we are considering satisfy the strong separation condition and let μ_u, ν_u and ρ_u^q be defined as above. There exists a constant c such that for all $x \in \operatorname{supp}\mu_u$ and $r > 0$,*

$$c^{-1}\rho_u^q(B(x,r)) \le \mu_u(B(x,r))^q\,\nu_u(B(x,r))^{\zeta_{\mu_u,\nu_u}(q)} \le c\rho_u^q(B(x,r)).$$

Fix $q \in \mathbf{R}$, and for $m, n \in \mathbf{N}$ define ϵ_m, ζ_n and $\eta_{m,n}$ using the following equations.

$$\frac{\alpha_{\nu_u}(q)}{\alpha_{\nu_u}(q) - 1/m} := 1 + \epsilon_m, \quad \frac{\gamma_{\mu_u,\nu_u}(q)}{\gamma_{\mu_u,\nu_u}(q) + 1/n} := 1 - \zeta_n \quad \text{and} \quad (1+\epsilon_m)(1-\zeta_n) := 1 + \eta_{m,n}.$$

Suppose, by way of contradiction, that

$$\dim_{\mathrm{P}}(E(q)) > \alpha_{\nu_u}(q)\left(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q)\right) := f(q).$$

For $m \in \mathbf{N}$ set

$$E_m = \left\{x \in E(q) \,\middle|\, r^{\alpha_{\nu_u}(q) - 1/m} \le \nu_u(B(x,r)) \text{ for } r < \frac{1}{m}\right\}$$

Then $E_m \nearrow E(q)$ and thus there exists an $M \in \mathbf{N}$ such that for $m \ge M$, $\dim_{\mathrm{P}}(E_m) > f(q)$. Fix $m \ge M$ and choose $n \in \mathbf{N}$ such that $\eta_{m,n} \ge 0$. For $l \in \mathbf{N}$ set

$$F_l = \left\{x \in E_m \,\middle|\, \nu_u(B(x,r))^{\gamma_{\mu_u,\nu_u}(q)+1/n} \le \mu_u(B(x,r)) \le \nu_u(B(x,r))^{\gamma_{\mu_u,\nu_u}(q)-1/n} \text{ for } r < \frac{1}{l}\right\}.$$

Then since $F_l \nearrow E_m$ we can choose an $L \in \mathbf{N}$ such that for all $l \ge L$, $\dim_{\mathrm{P}}(F_l) > f(q)$. Fix $l \ge L$, then using results in [JP95] we are able to find a compact set C such that $C \subseteq F_l$ and $\dim_{\mathrm{P}}(C) > f(q)$. Let $\eta > 0$ be given and let U be an open δ-neighbourhood of C such that $\rho_u^q(U) \le \rho_u^q(C) + \eta \le \rho_u^q(E(q)) + \eta$. Now choose ϵ such that $0 < \epsilon < \operatorname{dist}\left\{C, \mathbf{R}^d \backslash U\right\}$.

The condition that $\dim_{\mathrm{P}}(C) > f(q)$ implies that there exists a δ such that $0 < \delta < \epsilon$ and a centred δ-packing $(B_i := B(x_i, r_i))_i$ of C such that

$$1 \le \sum_i (2r_i)^{f(q)}.$$

Thus,

$$\begin{aligned}
1 &\le 2^{f(q)} \sum_i r_i^{f(q)} = 2^{f(q)} \sum_i r_i^{\alpha_{\nu_u}(q)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))} \\
&\le 2^{f(q)} \sum_i \nu_u(B_i)^{(1+\epsilon_m)(q\gamma_{\mu_u,\nu_u}(q) + \zeta_{\mu_u,\nu_u}(q))} \\
&\le 2^{f(q)} \sum_i \nu_u(B_i)^{q(1+\epsilon_m)\gamma_{\mu_u,\nu_u}(q)}\,\nu_u(B_i)^{\zeta_{\mu_u,\nu_u}(q)(1+\epsilon_m)} \\
&\le 2^{f(q)} \sum_i \mu_u(B_i)^{q(1+\epsilon_m)(1-\zeta_n)}\,\nu_u(B_i)^{\zeta_{\mu_u,\nu_u}(q)(1+\epsilon_m)} \\
&\le c2^{f(q)} \sum_i \rho_u^q(B_i)\,\mu_u(B_i)^{q\eta_{m,n}}\,\nu_u(B_i)^{\epsilon_m\zeta_{\mu_u,\nu_u}(q)} \\
&\le c2^{f(q)} \sum_i \rho_u^q(B_i) \le c2^{f(q)} \rho_u^q\left(\bigcup_i B_i\right) \le c2^{f(q)} \rho_u^q(U) \\
&\le c2^{f(q)}\left(\rho_u^q(E(q)) + \eta\right).
\end{aligned}$$

Letting $\eta \searrow 0$ yields a contradiction since $\rho_u^q(E(q)) = 0$. ∎

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